UNIVERSITÉ MOHAMMED V – AGDAL
FACULTÉ DES SCIENCES
Rabat

N° d'ordre : CT19

THÈSE DE DOCTORAT

Présentée par

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En cotutelle avec l'Aix-Marseille Université

Discipline : Mathématiques
Spécialité : Analyse

Les vecteurs cycliques dans des espaces de fonctions analytiques

Soutenue le 28 juin 2013

Devant le jury

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Cyclic vectors in some spaces of analytic functions

A Dissertation
Presented for the Doctor of Philosophy Degree
The University of Aix-Marseille and
the University of Mohammed V-Agdal, Rabat

Abdelouahab Hanine

4 juillet 2013

Remerciements

Les travaux présentés dans cette thèse ont été effectués au sein du Laboratoire d'Analyse et Applications de la faculté des Sciences de Rabat, sous la direction de Monsieur Omar Ei-Fallah, en colaboration avec le Laboratoire d'Analyse, Topologie et Probabilité du Centre Mathématiques et informatique d'Aix-Marseille Université, sous la direction de Monsieur Alexander Borichev.

Je suis très reconnaissant envers mes encadrants Monsieur le Professeur Omar El-Fallah, Professeur à la faculté des sciences Rabat, et Monsieur le Professeur Alexander Borichev Professeur d'Aix Marseille Université d'avoir accepter de m'encadrer et diriger cette thèse. Leurs soutiens, disponibilité et encouragements m'ont permis de bien mener ce travail.

Je tiens à remercier vivement Monsieur le Professeur John Conway pour l'honneur et l'amabilité qu'il me fait en présidant le jury.

Je remercie chaleuresement Monsieur le Professeur El Hassan Zerouali, Professeur à la faculté des sciences Rabat, d'avoir accepter d'examiner ma thèse et de prendre part au jury.

Je tiens à exprimer ma reconnaissance à Azeddine Baalal et Pascal Thomas d'avoir accepté de juger ce travail, pour les remarques qu'ils m'ont faites et pour leur efficacité.

Je tiens à exprimer ma gratitude aux Professeurs, Karim Kellay, Alfonso Montès-Rodriguès et El Hassan Youssfi qui ont accepté de faire partie de mon jury.

Je remercie chaleureusement toutes les personnes du département de mathématiques de la faculté des sciences de Rabat et du Laboratoire d'Analyse, Topologie et Probabilité du Centre Mathématiques et informatique d'Aix-Marseille Université qui m'ont toujours reçue avec beaucoup de gentillesse lorsque j'avais des problèmes d'ordre administratif et informatique.

Je remercie vivement tous les membres du Laboratoire d'Analyse et Applications et du Centre Mathématiques et Informatique que j'ai côtoyés tout au long de la préparation de cette thèse pour leur soutien moral et leur sympathie.

Je suis très reconnaissant envers mes parents, ma famille et mes amis pour leur dévouement et encouragements. Qu'ils trouve ici l'expression de mes meilleurs remerciements.

Abstract

In this thesis, we study the cyclicity problem in some spaces of analytic functions on the open unit disc. We focus our attention on Korenblum type spaces and on weighted Bergman type spaces.

First, we use the technique of premeasures, introduced and developed by Korenblum in the 1970-s and the 1980-s, to give a characterization of cyclic functions in the Korenblum type spaces $\mathcal{A}_\Lambda^{-\infty}$. In particular, we give a positive answer to a conjecture by Deninger. Second, we use the so called resolvent transform method to study the cyclicity of the one point mass singular inner function $S(z) = e^{-\frac{1+z}{1-z}}$ in weighted Bergman type spaces, especially with weights depending on the distance to a subset of the unit circle.

Key words and phrases : Weighted Bergman space, cyclic function, singular inner function, generalized Phragmen-Lindelof principle, resolvent method. Korenblum type space, premeasure, Lambda-Carleson set, cyclic function, Lambda-singular part of premeasure.

Résumé de la thèse

Cette thèse est consacrée à l'étude du problème de la cyclicité dans certains espaces de fonctions analytiques. Nous nous intéressons aux espaces de Bergman et aux espaces de type Korenblum. La caractérisation des fonctions cycliques dans les espaces de fonctions analytiques est un problème d'approximation polynomiale.

On désignera par $\mathbb{D}$ le disque unité ouvert dans le plan complexe $\mathbb{C}$ et par $\mathbb{T}$ le cercle unité de $\mathbb{C}$. L'ensemble des fonctions holomorphes sur $\mathbb{D}$ sera noté $\mathrm{Hol}(\mathbb{D})$.

Soit X un espace vectoriel topologique de fonctions analytiques sur $\mathbb{D}$. Soit M_z l'opérateur de multiplication par la variable z défini par

$$M_z(f) = zf, \quad f \in X.$$

On suppose dans la suite de cette section, que M_z définit un opérateur borné sur X. On dit qu'un sous-espace fermé $\mathcal{M}$ de X (espace de Banach) est invariant par M_z (ou z-invariant) si $z\mathcal{M} \subset \mathcal{M}$. Pour une fonction $f \in X$, nous notons $[f]_X$ le plus petit sous-espace fermé de X contenant f et invariant par M_z. Il est alors clair que

$$[f]_X := \overline{\{p(z)f(z) : p(z) \quad \text{polynôme}\}}^X,$$

(on considère la fermeture dans X).

Une fonction $f \in X$ est dite cyclique dans X lorsque $[f]_X = X$.

Supposons que les fonctionnelles d'évaluations sur X définies par :

$$f \mapsto f(z), \quad z \in \mathbb{D},$$

sont bornées. Il est alors clair que pour qu'une fonction f soit cyclique dans X, il faut que f n'ait pas de zéros dans $\mathbb{D}$. Cette condition n'est pas suffisante en général. En effet, si $X = H^2$ l'espace de Hardy, alors d'après le théorème de Beurling [4], $f \in H^2$ est cyclique si et seulement si f est extérieure, c'est-à-dire ils existent $h \in L^1(\mathbb{T})$ et $|\lambda| = 1$ tels que

$$f(z) = \lambda \exp\Big(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} h(e^{it})\, dt\Big).$$

Espaces de type Korenblum

Définitions et généralités

Dans la suite, le majorant Λ désigne toujours une fonction positive, différentiable, strictement décroissante et convexe sur (0,1] telle que :

1. $\Lambda(0) = +\infty$
2. $t\Lambda(t)$ est une fonction continue, strictement croissante et concave sur $[0,1]$ et $t\Lambda(t) \to 0$ quand $t \to 0$.
3. Il existe $\alpha \in (0,1)$ tel que la fonction $t^\alpha \Lambda(t)$ est strictement croissante.
4. il existe $C > 0$, tel que pour tout $t \in (0,1)$ on a

$$\Lambda(t^2) \le C\Lambda(t). \tag{0.1}$$

Nous pouvons prendre comme exemples : $\Lambda(t) = \log^+ \log^+(1/t)$, $\Lambda(t) = (\log(1/t))^p$, $p > 0$.

L'espace de type Korenblum associé au poids Λ est donné par

$$\mathcal{A}_\Lambda^{-\infty} = \cup_{c>0} \mathcal{A}_\Lambda^{-c} = \bigcup_{c>0} \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \le \exp(c\Lambda(1-|z|)) \Big\}.$$

Dans ce travail nous étudions la cyclicité des éléments de $\mathcal{A}_\Lambda^{-\infty}$. Muni de la topologie limite inductive et de la multiplication ponctuelle l'espace $\mathcal{A}_\Lambda^{-\infty}$ devient une algèbre de Fréchet. Notons que les polynômes sont denses dans $\mathcal{A}_\Lambda^{-\infty}$, et que M_z et les évaluations sont continues sur $\mathcal{A}_\Lambda^{-\infty}$.

Prémesures Λ-bornées

B. Korenblum a étudié les prémesures Λ-bornées pour $\Lambda(t) = \log(\frac{1}{t})$. Dans cette section, nous étendons les résultats des deux papiers de B. Korenblum [31, 32] (voir aussi [24, Chapitre 7]) aux prémesures Λ-bornées, pour une fonction Λ vérifiant les conditions ci-dessus.

On note $\mathcal{B}(\mathbb{T})$ l'ensemble de tous les arcs ouverts, fermés et semi-ouverts de $\mathbb{T}$. Par convention on suppose aussi que $\emptyset, \mathbb{T} \in \mathcal{B}(\mathbb{T})$.

Définition 1 *Une prémesure μ est une fonction réelle définie sur $\mathcal{B}(\mathbb{T})$ est vérifiant*

1. $\mu(\mathbb{T}) = 0$
2. $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$ *pour tout* $I_1, I_2 \in \mathcal{B}(\mathbb{T})$ *avec* $I_1 \cap I_2 = \emptyset$ *et* $I_1 \cup I_2 \in \mathcal{B}(\mathbb{T})$
3. $\lim_{n\to+\infty} \mu(I_n) = 0$, *pour toute suite décroissante* $(I_n)_n \in \mathcal{B}(\mathbb{T})$ *avec* $\bigcap_n I_n = \emptyset$

Dans la suite, nous noterons $|I|$ la mesure de Lebesgue normalisée de I.

Définition 2 *Une prémesure μ est dite Λ-bornée s'il existe une constante positive $C \ge 0$ telle que pour tout arc $I \subset \mathbb{T}$,*

$$\mu(I) \le C|I|\Lambda(|I|).$$

On note B_Λ^+ l'ensemble des prémesures Λ-bornées, et par $\|\mu\|^+$ la plus petite constante $C > 0$ vérifiant la condition :

$$\mu(I) \le C|I|\Lambda(|I|).$$

Soit F un sous-ensemble fermé non vide du cercle unité $\mathbb{T}$. L'entropie de F associée à Λ (Λ-entropie) est définie par :

$$Entr_\Lambda(F) = \sum_n |I_n|\Lambda(|I_n|),$$

où $\{I_n\}_n$ sont les composantes connexes de $\mathbb{T} \setminus F$, et $|I|$ désigne la mesure de Lebesgue normalisée de l'arc I. On pose $Entr_\Lambda(\emptyset) = 0$.

Un ensemble F fermé non vide est dit ensemble Λ-Carleson s'il est de mesure de Lebesgue nulle ($|F| = 0$) et $Entr_\Lambda(F) < +\infty$.

On note par $\mathcal{C}_\Lambda$ l'ensemble de tous les ensembles Λ-Carleson et par $\mathcal{B}_\Lambda$ l'ensemble de tous les boréliens de $\mathbb{T}$ ($B \subset \mathbb{T}$) tels que $\overline{B} \in \mathcal{C}_\Lambda$.

Nous pouvons maintenant introduire la notion de mesure Λ-singulière.

Définition 3 *Une fonction $\sigma : \mathcal{B}_\Lambda \to \mathbb{R}$ est dite mesure Λ-singulière si*

1. $\sigma\big|_F$ *s'étend en une mesure de Borel finie sur $\mathbb{T}$, pour tout $F \in \mathcal{C}_\Lambda$, ($\sigma\big|_F(E) = \sigma(E \cap F)$) pour tout fermé de $\mathbb{T}$.*
2. *Il existe $C > 0$ tel que*

$$|\sigma(F)| \leq C Entr_\Lambda(F)$$

pour tout $F \in \mathcal{C}_\Lambda$.

Étant donnée μ une prémesure Λ-bornée, on dénote

$$\mu_s(F) = -\sum_n \mu(I_n), \tag{0.2}$$

où $F \in \mathcal{C}_\Lambda$ et les $\{I_n\}_n$ sont les arcs complémentaires de F dans le cercle $\mathbb{T}$. En utilisant le même argument que dans [31, Théorème 6], on peut voir que μ_s s'étend en une mesure Λ-singulière sur $\mathcal{B}_\Lambda$. La mesure μ_s est appelée la partie Λ-singulière de μ.

Nous allons introduire maintenant la notion de prémesure Λ-absolument continue, qui jouera un rôle important dans la caractérisation des fonctions cycliques de $\mathcal{A}_\Lambda^{-\infty}$.

Définition 4 *Une prémesure Λ-bornée est dite Λ-absolument continue s'il existe une suite de prémesures Λ-bornées $(\mu_n)_n$ telle que :*

1. $(\mu + \mu_n) \in B_\Lambda^+$ *et* $\sup_n \|\mu + \mu_n\|^+ < \infty$.
2. $\sup_I |(\mu + \mu_n)(I)| \to 0$ *quand $n \to +\infty$.*

Dans le cas des mesures, μ est absolument continue si et seulement si μ n'a pas de partie singulière. Ce résultat s'étend au cas des prémesures Λ-bornées, nous obtenons le résultat suivant.

Théorème 1 *Une prémesure $\mu \in B_\Lambda^+$ est Λ-absolument continue si et seulement si $\mu_s \equiv 0$.*

La preuve est détaillée dans [21, Section 2.8]

Fonctions harmoniques à croissance contrôlé par Λ

Rappelons qu'une fonction harmonique bornée sur $\mathbb{D}$ peut être représentée par l'intégrale de Poisson d'une fonction définie sur $\mathbb{T}$. Dans le théorème suivant nous montrons qu'une large classe de fonctions harmoniques à valeurs réelles dans le disque unité $\mathbb{D}$ peut être représentée par les intégrales de Poisson de prémesures Λ-bornées.

Soit $\mu \in B_\Lambda^+$, nous notons $P[\mu]$ le noyau de Poisson de μ,

$$P[\mu](z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\mu(\theta),$$

et nous définissons l'intégrale en terme de prémesures Λ-bornées par :

$$P[\mu](z) := -\int_0^{2\pi} \Big(\frac{\partial}{\partial\theta}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\Big)\overset{\curlywedge}{\mu}(\theta)\, d\theta,$$

telle que

$$\overset{\curlywedge}{\mu}(\theta) = \mu(I_\theta),$$

où

$$I_\theta = \big\{\xi \in \mathbb{T} : 0 \leq \arg \xi < \theta\big\}.$$

Théorème 2 *Soit h une fonction harmonique sur $\mathbb{D}$, à valeurs dans $\mathbb{R}$ avec $h(0)=0$ telle que*

$$h(z) = O(\Lambda(1-|z|)), \qquad |z| \to 1,\ z \in \mathbb{D}.$$

Les assertions suivantes sont vérifiées :

1. *Pour tout arc I du cercle unité $\mathbb{T}$, la limite suivante existe*

$$\mu(I) = \lim_{r\to 1^-} \mu_r(I) = \lim_{r\to 1^-} \int_I h(r\xi)\, |d\xi| < \infty.$$

2. *μ est une prémesure Λ-bornée.*
3. *La foction h admet la représentation suivante (l'intégrale de Poisson de prémesure μ) :*

$$h(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\mu(\theta), \qquad z \in \mathbb{D}.$$

Réciproquement, nous montrons que le noyau de Poisson d'une prémesure Λ-bornée est une fonction harmonique à croissance contrôlée par Λ, plus précisément

$$P[\mu](z) \leq 10\|\mu\|_\Lambda^+ \Lambda(1-|z|), \qquad z \in \mathbb{D}.$$

Grâce à ce résultat, on peut représenter les fonctions de $\mathcal{A}_\Lambda^{-\infty}$ qui ne s'annulent pas dans $\mathbb{D}$ comme les intégrales de Poisson de prémesures Λ-bornées. En effet, si $f \in \mathcal{A}_\Lambda^{-\infty}$, $f(0)=1$, il existe une prémesure Λ-bornée $\mu_f(I) := \mu(I) = \lim\limits_{r\to 1^-} \mu_r(I) = \lim\limits_{r\to 1^-} \int_I \log|f(r\xi)|\,|d\xi|$, $I \in \mathcal{B}(\mathbb{T})$ (voir le théorème 2) telle que :

$$f(z) = f_\mu(z) := \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta). \tag{0.3}$$

Cyclicité

Le résultat suivant découle du théorème 1.

Théorème 3 *Soit $f \in \mathcal{A}_\Lambda^{-\infty}$ une fonction sans zéros sur $\mathbb{D}$ telle que $f(0) = 1$. Si $(\mu_f)_s \equiv 0$, alors f est cyclique dans $\mathcal{A}_\Lambda^{-\infty}$.*

Dans ce qui suit, nous traitons la réciproque du théorème 3. Nous établissons un résultat principal valable pour deux type de croissance de la fonction Λ. Plus précisément, nous considérons les deux cas suivants :

Cas 1. Nous supposons que le majorant Λ vérifie la condition (C1), définie par :

$$\text{pour toute } c > 0,\ x \mapsto \exp\big[c\Lambda(1/x)\big] \text{ est une fonction concave pour } x \text{ grand.} \tag{C1}$$

Exemple du majorant Λ qui vérifie la condition (C1) :

$$(\log(1/x))^p, \quad 0 < p < 1, \quad \text{and} \quad \log(\log(1/x)), \quad x \to 0.$$

Cas 2. Nous supposons que le majorant Λ vérifie la condition (C2), définie par :

$$\lim_{t \to 0} \frac{\Lambda(t)}{\log(1/t)} = \infty. \tag{C2}$$

Exemple du majorant Λ qui vérifie la condition (C2) :

$$(\log(1/x))^p, \quad p > 1.$$

Nous obtenons le résultat suivant :

Théorème 4 *Soit $\mu \in B_\Lambda^+$. Alors la fonction f_μ est cyclique dans $\mathcal{A}_\Lambda^{-\infty}$ si et seulement si $\mu_s \equiv 0$.*

Pour la démonstration de ce théorème, nous distinguons trois cas :

Si Λ vérifie la condition (C1), nous utilisons théorème de Shirokov (voir [46, Théorème 9, pp. 137,139]).

Si $\Lambda(t) = \log(\frac{1}{t})$ c'est le théorème de B. Korenblum [32, Théorème 3.1].

Si Λ vérifie la condition (C2), nous utilisons le théorème de Taylor et Williams (voir [11, Théorème 5.3]).

Remarquons que dans le cas 1, nous considérons des poids à croissance lente par rapport au poids de Korenblum ($\Lambda(x) = \log(1/x)$) et dans le cas 2, la croissance de Λ est plus rapide que celle de Korenblum.

Le théorème 4, donne une réponse positive à une conjecture énoncée par C. Deninger [14, Conjecture 42].

Nous donnons maintenant deux exemples qui montrent comment la cyclicité d'une fonction fixée change en changeant le majorant Λ dans l'espace $\mathcal{A}_\Lambda^{-\infty}$.

Exemple 1 *Soit* $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, *et soit* $0 < \alpha_0 < 1$. *Il existe une fonction singulière intérieure* S_μ *telle que*

$$S_\mu \text{ est cyclique dans } \mathcal{A}^{-\infty}_{\Lambda_\alpha} \iff \alpha > \alpha_0.$$

Exemple 2 *Soit* $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, *et soit* $0 < \alpha_0 < 1$. *Il existe une prémesure* μ *telle que* μ_s *est infinie et*

$$f_\mu \text{ est cyclique dans } \mathcal{A}^{-\infty}_{\Lambda_\alpha} \iff \alpha > \alpha_0,$$

où f_μ *est définie par la formule dans* (0.3).

Il faut remarquer que le sous-espace $[f_\mu]_{\mathcal{A}^{-\infty}_{\Lambda_\alpha}}$, $\alpha \le \alpha_0$, est un sous-espace fermé non trivial de $\mathcal{A}^{-\infty}_{\Lambda_\alpha}$. De plus, il ne contient aucune fonction non nulle de la classe de Nevanlinna $N := \{f \in \mathrm{Hol}(\mathbb{D}) : \sup_{0\le r<1} \int_{\mathbb{T}} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty\}$.

Cyclicité dans les espaces de type Bergman

Dans cette partie, nous étudions la cyclicité de la fonction intérieure singulière $S(z) = e^{-\frac{1+z}{1-z}}$ dans les espaces de type Bergman.

Étant donnée une fonction positive continue strictement décroissante Λ sur $(0, 1]$ et $E \subset \mathbb{T} = \partial\mathbb{D}$, on note $\mathcal{B}^\infty_{\Lambda,E}$ l'espace des fonctions analytiques f sur $\mathbb{D}$ telles que

$$\|f\|_{\Lambda,E,\infty} = \sup_{z\in\mathbb{D}} |f(z)|e^{-\Lambda(\mathrm{dist}(z,E))} < +\infty,$$

et par $\mathcal{B}^{\infty,0}_{\Lambda,E}$ son sous-espace séparable

$$\mathcal{B}^{\infty,0}_{\Lambda,E} = \big\{f \in \mathcal{B}^\infty_{\Lambda,E} : \lim_{\mathrm{dist}(z,E)\to 0} |f(z)|e^{-\Lambda(\mathrm{dist}(z,E))} = 0\big\}.$$

De même, en intégrant par rapport à la mesure de Lebesgue sur le disque $\mathbb{D}$, nous définissons les espaces $\mathcal{B}^p_{\Lambda,E}$, $1 \le p < \infty$:

$$\mathcal{B}^p_{\Lambda,E} = \Big\{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|^p_{\Lambda,E,p} = \int_{\mathbb{D}} |f(z)|^p e^{-p\Lambda(\mathrm{dist}(z,E))} < +\infty\Big\}.$$

Dans le cas où l'ensemble $E = \mathbb{T}$, nous utilisons les notations $\mathcal{B}^\infty_\Lambda$, $\mathcal{B}^{\infty,0}_\Lambda$, $\mathcal{B}^p_\Lambda$ à la place de $\mathcal{B}^\infty_{\Lambda,E}$, $\mathcal{B}^{\infty,0}_{\Lambda,E}$, $\mathcal{B}^p_{\Lambda,E}$. Remarquons que, si $\lim_{t\to 0^+} \Lambda(t) < \infty$, $\mathcal{B}^\infty_\Lambda = H^\infty$, $\mathcal{B}^{\infty,0}_\Lambda = \{0\}$, $\mathcal{B}^p_\Lambda = \mathcal{B}^p_0$.

Pour une suite de nombres positifs $w = \{w_n\}_n$ telle que $|\log w_n| = o(n)$, on note

$$H^2_w = \Big\{f(z) = \sum_{n\ge 0} a_n z^n, \qquad \sum_{n\ge 0} \frac{|a_n|^2}{w_n} < \infty\Big\}.$$

Il est bien connu que si la suite w_n est log-convexe alors $H^2_w = \mathcal{B}^2_\Lambda$ (il existe Λ telle que

$$w_n \asymp \Big(\int_0^1 r^{2n+1} e^{-2\Lambda(1-r)}\,dr\Big)^{-1}$$

voir [8, Proposition 4.1]).

En 1964 Beurling [4] a étudié la cyclicité de S dans l'espace $\bigcup_{k\geq 1} H^2_{w^k}$ (muni de la topologie limite inductive usuelle). Il démontre grâce au théorème d'approximation de Bernstein, sous des conditions de régularité sur le poids w, que toute fonction de $\bigcup_{k\geq 1} H^2_{w^k}$ qui ne s'annule pas dans $\mathbb{D}$ est cyclique dans $\bigcup_{k\geq 1} H^2_{w^k}$ si et seulement si S est cyclique dans $\bigcup_{k\geq 1} H^2_{w^k}$ si et seulement si

$$\sum_{n\geq 1} \frac{\log w_n}{n^{3/2}} = \infty.$$

En 1974 N. Nikolski (voir [39, Section 2.6]) a montré que si $\liminf_{t\to 0} \frac{\Lambda(t)}{\log 1/t} > 0$, et si la fonction intérieure S est cyclique dans $\mathcal{B}^\infty_\Lambda$, alors

$$\int_0 \sqrt{\frac{\Lambda(t)}{t}}dt = \infty. \tag{0.4}$$

Pour la réciproque, il a prouvé que si

$$\text{la fonction } t \mapsto t\Lambda'(t) \text{ est strictement croissante,} \tag{0.5}$$

et Λ vérifie la condition (0.4), alors S est cyclique dans $\mathcal{B}^\infty_\Lambda$. La preuve de ce résultat utilise un théorème de quasi-analyticité. Pour cela, la condition (0.5) qui traduit en quelque sorte une condition de convexité sur Λ est indispensable dans la démonstration.

Dans la première partie de ce paragraphe nous prouvons les deux résultats suivants :

Théorème 5 *Soit Λ une fonction continue, positive et strictement décroissante sur $(0,1]$. Alors $S(z) = e^{-\frac{1+z}{1-z}}$ est cyclique dans $\mathcal{B}^{\infty,0}_\Lambda$ si et seulement si Λ vérifie la condition (0.4).*

Théorème 6 *Soit $1 \leq p < \infty$, et soit Λ une fonction continue, positive et strictement décroissante sur $(0,1]$. Alors $S(z) = e^{-\frac{1+z}{1-z}}$ est cyclique dans $\mathcal{B}^p_\Lambda$ si et seulement si Λ vérifie la condition (0.4).*

En fait, nous avons pu éliminer les conditions de régularité imposées par N. Nikolski sur la fonction Λ en utilisant "la méthode de la résolvante" exposée, par exemple dans [15, 8]. Cette technique a été introduite par Carleman et Gelfand ; ensuite Y. Domar l'a utilisé pour étudier les idéaux fermés dans certaines algèbres de Banach.

En 1986 Gevorkyan et Shamoyan [20] ont montré, sous quelques conditions de régularité sur la fonction Λ, que la condition

$$\int_0 \Lambda(t)\,dt = \infty, \tag{0.6}$$

est une condition nécessaire et suffisante pour la cyclicité de la fonction S dans l'espace $\mathcal{B}^{\infty,0}_{\Lambda,\{1\}}$. Récemment, El-Fallah, Kellay, et Seip ont amélioré les résultats dû a Beurling et Nikolski ainsi que le résultat de Gevorkyan et Shamoyan. Leur démonstration est basée sur le théorème de la couronne. Pour plus de détails voir [17].

Notons que notre méthode s'applique directement dans le cas où E est un arc fermé contenant le point 1. De plus, pour une fonction Λ suffisamment régulière nous obtenons des conditions nécessaires et suffisantes en terme du comportement de E au voisinage du point 1.

Dans la deuxième partie de ce paragraphe, nous étudions la cyclicité de S dans $\mathcal{B}^{\infty,0}_{\Lambda,E}$ en fonction de Λ et de E. C'est-à-dire, pour une Λ suffisamment régulière nous obtenons des conditions nécessaires et suffisantes de la cyclicité en fonction de E. Plus précisément, pour une fonction Λ définie par $\Lambda(t) = \frac{1}{tw(t)^2}$ avec w est une fonction suffisamment régulière, nous avons le résultat suivant :

Théorème 7 *Soit Λ la fonction définie ci-dessus. Alors S est cyclique dans $\mathcal{B}^{\infty,0}_{\Lambda,E}$ si et seulement si l'une des trois quantités suivantes est infinie :*

$$\int_{e^{it}\in E} \frac{dt}{|t|w(|t|)}, \quad \int_0 \frac{dt}{|t|w^2(|t|)}, \quad \sum_n \frac{1}{w(b_n)^2} \log\Big[1 + \Big(1 - \frac{a_n}{b_n}\Big) w(b_n)\Big], \tag{0.7}$$

où (e^{ia_n}, e^{ib_n}), (e^{-ib_n}, e^{-ia_n}), $0 < a_n < b_n$. sont les arcs complémentaires de E dans $\mathbb{T}$.

La preuve est détaillée dans [6, Section 6].

Dans le corollaire suivant nous donnons deux exemples où E est un ensemble dénombrable

Corollaire 1 *Soit Λ la fonction définie ci-dessus. Si $E = \{\exp(i \cdot 2^{-n})\}_{n\geq 1} \cup \{1\}$, alors S est cyclique dans $\mathcal{B}^{\infty,0}_{\Lambda,E}$ si et seulement si*

$$\sum_{n\geq 1} \frac{\log w(2^{-n})}{w(2^{-n})^2} = +\infty\,;$$

si $E = \{\exp(i \cdot 2^{-2^n})\}_{n\geq 1} \cup \{1\}$, alors S est cyclique dans $\mathcal{B}^{\infty,0}_{\Lambda,E}$ si et seulement si

$$\int_0 \frac{dt}{tw(t)^2} = +\infty,$$

et on retrouve la condition de Gevorkyan–Shamoyan pour $E = \{1\}$.

Dans la suite, nous donnons encore deux applications du critère général (0.7). Dans la première application, nous obtenons un résultat d'interpolation entre la condition de Nikolski et de Gevorkyan–Shamoyan. Pour cela, nous introduisons la condition suivante

$$\int_0 \frac{\Lambda(t)^{1-\beta}}{t^\beta}\, dt = +\infty, \qquad 0 \leq \beta \leq \frac{1}{2}. \tag{C_β}$$

Remarquons que $C_{1/2}$ est la condition de Nikolski ($\int_0 \sqrt{\Lambda(t)/t}\, dt = +\infty$) et que C_0 est la condition de Gevorkyan–Shamoyan ($\int_0 \Lambda(t)\, dt = +\infty$).

Pour

$$\Lambda_\alpha(t) = \frac{1}{t \log^\alpha(1/t)}$$

on a

$$\Lambda_\alpha \in (C_\beta) \iff \alpha(1-\beta) \leq 1.$$

Théorème 8 *Soit* $0 \leq \beta \leq 1/2$, $a_n = \exp(-n^{1-\beta})$, $n \geq 1$, $E_\beta = \{e^{ia_n}\}_{n\geq 1} \cup \{1\}$. *La fonction* $S(z) = e^{-\frac{1+z}{1-z}}$ *est cyclique dans* $\mathcal{B}^{\infty,0}_{\Lambda_\alpha,E_\beta}$ *si et seulement si* $\Lambda_\alpha \in (C_\beta)$.

Le théorème 7 peut aussi être appliqué aux ensembles parfaits comme le montre le théorème suivant. Nous notons κ la dimension de Hausdorff de l'ensemble triadique de Cantor E (voir [18, Section 1.5]), $\kappa = \frac{\log 2}{\log 3}$.

Théorème 9 *Soit* E *un ensemble triadique de Cantor. La fonction* $S(z) = e^{-\frac{1+z}{1-z}}$ *est cyclique dans* $\mathcal{B}^{\infty,0}_{\Lambda_\alpha,E}$ *si et seulement si*

$$\alpha \leq \frac{1}{1-\frac{\kappa}{2}}.$$

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Chapitre 1

Introduction

Throughout this manuscript, we use the following notations : given two functions f and g defined on Δ we write $g \asymp f$ or $g \lesssim f$ if for some $0 < c_1 \leq c_2 < +\infty$ we have $c_1 f \leq g \leq c_2 f$, respectively $g \leq c_2 f$ on Δ.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Suppose that X is a topological vector space of analytic functions on $\mathbb{D}$, with the property that $zf \in X$ whenever $f \in X$. Multiplication by z (shift operator) is thus an operator on X, and if X is a Banach space, then it is automatically a bounded operator on the space X. A closed subspace $M \subset X$ (Banach space) is said to be invariant (or z-invariant) provided that $zM \subset M$. For a function $f \in X$, the closed linear span in X of all polynomial multiples of f is a z-invariant subspace denoted by $[f]_X$; it is also the smallest closed z-invariant subspace of X which contains f. A function f in X is said to be cyclic (or weakly invertible) in X if $[f]_X = X$. If the polynomials are dense in X, an equivalent condition is that $1 \in [f]_X$. For some information on cyclic functions see [7] and the references therein.

If the point evaluation functionals,

$$f \mapsto f(z), \quad z \in \mathbb{D},$$

are bounded, then an immediate necessary condition for the function f to be cyclic is that the function f have no zeros on $\mathbb{D}$. In general, it is a difficult problem to give necessary and sufficient conditions for cyclicity.

1.1 Hardy spaces

In this section, we will define the Hardy spaces and the Nevanlinna class, we introduce also some notations and properties of these classes of functions that are going to be needed later on. The books [19], [16], and [29] are excellent sources of information about Hardy spaces.

1.1.1 H^p-Functions

Definitions and factorization.

In this section we are concerned with the multiplicative structure of the Hardy spaces, in that we want to factorize a general Hardy class function as the product of two somewhat simpler functions, an inner factor and an outer factor.

Let $0 < p < \infty$ and let $f(z)$ be an analytic function on $\mathbb{D}$. We say that $f \in H^p := H^p(\mathbb{D})$ (the Hardy space) if the integrals

$$\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} := \|f\|_{H^p}^p,$$

are bounded for $r < 1$. If $p = \infty$, we say that $f \in H^\infty$, the algebra of bounded analytic function on $\mathbb{D}$ if

$$\sup_{z\in\mathbb{D}} |f(z)| := \|f\|_\infty < \infty.$$

Suppose that f is a non-zero function of the class H^1 on the unit disc. Then f has non-tangential limits at almost every point of the unit circle $\mathbb{T}$:

$$f^*(e^{i\theta}) := \lim_{z\to e^{i\theta}} f(z),$$

where the limit is taken as $z \in \mathbb{D}$ approaches the boundary point $e^{i\theta}$ within a sector defined by $|arg(e^{i\theta} - z)| \leq \alpha$, for any constant $\alpha < \pi/2$. Furthermore, the function $\log |f^*(e^{i\theta})|$ is Lebesgue integrable.

Definition 1.1 *An inner function is an* $\mathrm{H}^\infty(\mathbb{D})$ *function that has unit modulus almost everywhere on* $\mathbb{T}$*. An outer function is a function* $F \in H^1$ *which can be written in the form*

$$F(re^{i\theta}) = \alpha \exp\Big(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\phi(e^{it})\,dt\Big), \tag{1.1}$$

where ϕ *is a real-valued integrable function and* $|\alpha| = 1$*.*

Proposition 1.2 *Let* f *be an outer function, satisfying* (1.1)*. Then* $\log |f^*(e^{i\theta})| = \phi(e^{i\theta})$ *almost everywhere.*

The following theorem gives us an important class of inner functions.

Theorem 1.3 *Let* S *be an inner function without zeros on* $\mathbb{D}$*, and suppose that* $S(0)$ *is positive. Then there is a unique positive measure* μ*, singular with respect to Lebesgue measure on the unit circle, and a constant* α *of modulus* 1*, such that*

$$S(re^{i\theta}) := S_\mu(re^{i\theta}) = \alpha \exp\Big(-\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\,d\mu(t)\Big).$$

These functions S are called singular inner functions. An important example is the atomic singular inner function

$$S(z) = \exp\Big\{ -\frac{1+z}{1-z} \Big\},$$

with its measure μ concentrated at the point 1.

The representation $f(z) = e^{i\theta} S(z) F(z)$ of function without zeros on $\mathbb{D}$, is unique and is known as the canonical factorization of a zero-free H^p function.

1.1.2 Cyclic vectors

Theorem 1.4 (A. Beurling) [4].
Any closed shift-invariant subspace M of H^2 has the form $M = \theta H^2$, where θ is inner.

We may use Beurling's theorem to deduce a fairly user-friendly characterization of cyclic functions

Corollary 1.5 *$f \in H^2$ is cyclic if and only if it is an outer function.*

1.1.3 The Nevanlinna Class

An analytic function f in $\mathbb{D}$ is said to be in the Nevanlinna class, $f \in N$ if the integrals

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi},$$

are bounded for $r < 1$, where $\log^+ a = max\{\log a, 0\}$. It is clear that, $H^p \subset N$ for all $p > 0$, because $\log^+ |f(z)| \leq \frac{1}{p}|f(z)|^p$. It is known that functions in the Nevanlinna class are quotients of H^∞ functions, so each function $f \in N$ has a non-tangential limit $f^*(e^{i\theta})$ at almost every boundary point $e^{i\theta}$. Furthermore, given a zero-free function $f \in N$, there exists a unique outer function F and unique singular inner functions $S_\mu(z)$ and $S_\nu(z)$ with mutually singular associated measures such that (see [16, Thm. 2.9, p. 25])

$$f(z) = \frac{S_\mu(z)}{S_\nu(z)} F(z).$$

1.2 Bergman Spaces

in this section, we introduce some properties of Bergman spaces, and we will review some results and problems concerning the cyclicity problem in these spaces.

1.2.1 Definitions and properties

For $0 < p < \infty$, the Bergman space $A^p(\mathbb{D}) := A^p$ of the disc is the space of analytic functions in $\mathbb{D}$ for which

$$\|f\|_p^p := \|f\|_{A^p}^p = \frac{1}{\pi}\int\int_{\mathbb{D}} |f(z)|^p\,dxdy, \quad (z = x + iy).$$

Equipped with the above norm, A^p is a Banach space for $1 \leq p < \infty$. If $0 < p < 1$, the triangle inequality is replaced by the inequality $\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$ (a quasi-Banach space for $0 < p < 1$). In particular, A^p is always a linear metric space.

The following theorem asserts that functions in a Bergman space cannot grow too rapidly near the boundary.

Theorem 1.6 *The point evaluation is a bounded linear functional in each Bergman space A^p. More specifically, every function f in A^p satisfies the following estimate :*

$$|f(z)| \leq \|f\|_p \pi^{-\frac{1}{p}} \frac{1}{(1 - |z|)^{\frac{2}{p}}}, \quad z \in \mathbb{D}.$$

A^2 is a reproducing kernel Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int\int_{\mathbb{D}} f(z)\overline{g(z)}\,dxdy,$$

and its kernel is called the Bergman kernel, and denoted by $K(w, z)$.

Lemma 1.7 *The Bergman kernel for the open unit disc $\mathbb{D}$ is given by*

$$K(w, z) = \frac{1}{\pi}\frac{1}{(1 - w\bar{z})^2}, \quad \textit{for} \quad w, z \in \mathbb{D}.$$

Note also that the Bergman space A^p contains H^p as a dense subspace, and we have $f^*(e^{i\theta}) \in L^p(\mathbb{T})$, $f \in H^p$ (non-tangential limits exists at almost every boundary point for H^p functions). This is however not the case for $A^p(\mathbb{D})$. In fact, there is a function in it which fails to have radial limits at every point of $\mathbb{T}$. For example, we can take the function $f(z) = \sum\limits_{n=1}^{+\infty} z^{2^{2^n}} \in A^p \setminus N$ (see [12]) ; then $f(z)$ has no finite radial limits.

1.2.2 Factorization

We pass to the factorization theory in the Bergman space. Let M be a proper closed subspace of A^p invariant under multiplication by z. Denote by n the smallest nonnegative integer such that there exists a function $f \in M$ with $f^{(n)}(0) \neq 0$, and consider the following extremal problem

$$\sup\left\{\mathrm{Re} f^{(n)}(0) : \quad f \in M, \|f\|_{A^p} = 1\right\}.$$

Suppose that M is a cyclic invariant subspace (singly generated invariant subspace), and that the generator is $g \in A^p$. Then the above problem has a unique solution, which we denote by G and call the extremal solution (or canonical divisor) for the subspace M. It is proved in [2] (see also [24, Chapter 3]) that $M = [G]$, and that G have the so-called expansive multiplier property, or equivalently, the contractive divisibility property on M :

$$\left\| \frac{g}{G} \right\|_{A^p} \leq \|g\|_{A^p}, \quad g \in M.$$

Theorem 1.8 *Suppose that $0 < p < \infty$ and $f \in A^p$. Then there exists G (the solution to the extremal problem for $M = [f]$), and a cyclic vector $F \in A^p$ such that $f = GF$. Furthermore,*

$$\|F\|_{A^p} \leq \|f\|_{A^p}.$$

In the classical theory of Hardy spaces H^p, the inner-outer factorization is unique (up to a unimodular constant multiple of the inner factor). Unfortunately, the factorization in Bergman spaces A^p does not have such a strong uniqueness property (for the explication, see [24, Corollary 8.8]).

1.2.3 Cyclicity

Cyclic Vectors as Outer functions.
Boris Korenblum introduced in [34] the notion of outer function for Bergman spaces A^p in terms of domination and proved that a cyclic function in A^p necessarily is outer. We write $f \succ g$ (or $g \prec f$) and say that f dominates g in A^p if

$$\|fh\|_{A^p} \geq \|gh\|_{A^p}, \quad h \in H^\infty.$$

Note that for H^p-functions $f \succ g$ is equivalent to $|f(e^{i\theta})| \geq |g(e^{i\theta})|$ almost everywhere. The following definition is motivated by the properties of (Beurling) outer function.

Definition 1.9 *We say that a function $f \in A^p$ is A^p-outer if $|f(0)| \geq |g(0)|$ whenever $g \prec f$ in A^p .*

The following theorem was conjectured by Korenblum in [34], and later proved by Aleman, Richter, and Sundberg in [2].

Theorem 1.10 *Let $f \in A^p$ with $0 < p < \infty$. Then f is A^p-outer if and only if it is cyclic in A^p.*

Singular inner functions are sometimes cyclic.
A zero-free function f in the class $A^p \cap N$ ($f(z) = \frac{S_\mu(z)}{S_\nu(z)} F(z)$) is cyclic in A^p if and only if its singular inner function S_μ is cyclic,

$$S_\mu(z) = \exp\Big[-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \Big], \qquad z \in \mathbb{D},$$

i.e. if and only if its associated positive singular measure μ places no mass on any Carleson set (Beurling-Carleson sets). Carleson sets constitute a class of thin subsets of $\mathbb{T}$, they will be discussed below. The necessity of this Carleson set condition was proved by H. S. Shapiro in 1967 [42, Theorem 2], and the sufficiency was proved independently and using different arguments by B. Korenblum in 1977 [33] and J. Roberts in 1979 [40, Theorem 2].

Furthermore, H. S. Shapiro [43, Theorem 2] showed that there exists an absolute constant $C > 0$ such that

$$|S_\mu(z)| \geq \exp\Big(- \frac{C\omega_\mu(1-|z|)}{1-|z|}\Big),$$

where ω_μ is the modulus of continuity of μ defined by

$$\omega_\mu(t) = \sup\{\mu(I) : |I| < t, \quad I \quad \text{a sub-arc of } \mathbb{T}\}.$$

In particular, the singular inner functions with $\omega_\mu(t) = O\big(t\log(\frac{1}{t})\big)$ are cyclic in Bergman spaces A^p.

Invertibility and cyclicity.

A function f in a space X of analytic functions is said to be invertible if $1/f$ also belongs to X. In the classical theory of Hardy spaces, it is known that every invertible function in H^p is necessarily cyclic in H^p. This is also true in the $A^{-\infty}$ space (the Korenblum space, see definition 1.12 below) ; an invertible function f in $A^{-\infty}$,

$$|f(z)| \geq C(1-|z|)^c \quad z \in \mathbb{D}, \tag{1.2}$$

for some positive numbers C, c, is always cyclic in $A^{-\infty}$.

In the case of Bergman spaces, H. S. Shapiro (see [42]), posed the following question. Suppose that f satisfies (1.2). Is f cyclic in A^2 ? This question was settled in the negative by A. Borichev and H. Hedenmalm [7]. The construction involved first finding harmonic functions with special growth properties, and then forming the zero-free function obtained by harmonic conjugation plus exponentiation.

The following theorem due to H. S. Shapiro [42, 44], gives a slightly more restrictive sufficient condition for cyclicity

Theorem 1.11 *If $f \in A^q$ for some $q > p$, and if there are positive constants c and C such that $|f(z)| \geq C(1-|z|)^c$ for all $z \in \mathbb{D}$, then f is cyclic in A^p.*

So, it is natural to ask (see for example, Question 6 in [45], posed by A. L. Shields) if there exists any decreasing radial function $\phi(z) = \phi(|z|)$, with $\phi(r) \to 0, \quad \text{as} \quad r \to 1^-$ such that the conditions $f \in A^p$ and $|f(z)| \geq \phi(|z|)$ for $z \in \mathbb{D}$ would imply that f is cyclic in A^p. In [10] A. Borichev shows that the answer is affirmative. More specifically, the function

$$\phi(t) = \exp\left[- \Big(\log\frac{1}{1-t}\Big)^{1/(2+\alpha)}\right] \quad \alpha > 0,$$

has the desired property.

It remains an open problem to give an explicit characterization of the cyclic functions in Bergman spaces.

1.3 Korenblum spaces

In this section we will give a short introduction to Korenblum's work on generalized Nevanlinna theory. Furthermore, we review some results on the cyclicity problem in these spaces.

1.3.1 Definitions and properties

The classical representation and factorization theory due to R. Nevanlinna [38] is based on the application of the Poisson-Jensen formula to smaller disks $|z| \leq r < 1$ and on a subsequent transition to the limit as $r \to 1$ involving an application of the Riesz-Herglotz formula. The Riesz-Herglotz theorem does not directly applies to general classes of harmonic functions which makes difficult to extend Nevanlinna theory to these classes.

An important contribution was made here by Korenblum who introduced what is know now as "Korenblum spaces"

Definition 1.12 *The Korenblum space $A^{-\infty}$ is a topological algebra of analytic functions f in $\mathbb{D}$ that satisfy the following estimate*

$$|f(z)| \leq C_f \frac{1}{(1-|z|)^c}, \qquad C_f, c > 0,$$

i.e

$$A^{-\infty} = \cup_{c>0} A^{-c} = \bigcup_{c>0} \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \leq C_f \frac{1}{(1-|z|)^c} \Big\}.$$

Note that the class $A^{-\infty}$ is the smallest extension ring of algebra H^∞ invariant under differentiation. Furthermore, $A^{-\infty}$ contains (in fact, is the union of) the Bergman spaces A^p $(0 < p < \infty)$. Since the set of polynomials is dense in $A^{-\infty}$ it is readily seen that every invariant subspace for the operator M_z (multiplication by z) is a closed ideal in the algebra $A^{-\infty}$, and vice versa.

With the norm

$$\|f\|_{A^{-c}} = \sup_{z\in\mathbb{D}} (1-|z|)^c |f(z)| < \infty,$$

A^{-c} becomes a Banach space and for every $c_2 \geq c_1 > 0$, the inclusion $A^{-c_1} \hookrightarrow A^{-c_2}$ is continuous. The topology on

$$A^{-\infty} = \cup_{c>0} A^{-c},$$

is the locally-convex inductive limit topology, i.e. each of the inclusions $A^{-c} \hookrightarrow A^{-\infty}$ is continuous and the topology is the largest locally-convex topology with this property. A sequence $\{f_n\}_n \in A^{-\infty}$ converges to $f \in A^{-\infty}$ if and only if there exists $N > 0$ such that all f_n and f belong to A^{-N}, and $\lim_{n\to+\infty} \|f_n - f\|_{A^{-N}} = 0$.

1.3.2 Representation and canonical factorization

Let $\mathcal{B}(\mathbb{T})$ be the set of all (open, half-open and closed) arcs of $\mathbb{T}$ including all the single points and the empty set. The elements of $\mathcal{B}(\mathbb{T})$ will be called intervals.

Definition 1.13 *A real valued function defined on $\mathcal{B}(\mathbb{T})$ is called a premeasure if the following conditions hold :*

1. $\mu(\mathbb{T}) = 0$
2. $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$ *for every* $I_1,\ I_2 \in \mathcal{B}(\mathbb{T})$ *such that* $I_1 \cap I_2 = \emptyset$ *and* $I_1 \cup I_2 \in \mathcal{B}(\mathbb{T})$
3. $\lim_{n\to+\infty} \mu(I_n) = 0$ *for every sequence of embedded intervals,* $I_{n+1} \subset I_n$, $n \geq 1$, *such that* $\bigcap_n I_n = \emptyset$.

Definition 1.14 *A premeasure μ is called κ-bounded if there exists a positive constant C such that*

$$\mu(I) \leq C|I| \log \frac{2\pi e}{|I|}, \quad \textit{for all} \quad I \in \mathcal{B}(\mathbb{T}).$$

The set of all κ-bounded premeasures will be denoted B_κ^+.

Theorem 1.15 *[31] Let h be a real-valued harmonic function on the unit disk such that $h(0) = 0$ and*

$$h(z) = O(|\log(1 - |z|)|), \qquad |z| \to 1,\ z \in \mathbb{D}.$$

Then the following statements hold.

1. *For every open arc I of the unit circle $\mathbb{T}$ the following limit exists :*

$$\mu(I) = \lim_{r\to1^-} \mu_r(I) = \lim_{r\to1^-} \int_I h(r\xi)\,|d\xi| < \infty.$$

2. *μ is a κ-bounded premeasure.*
3. *The function h is the Poisson integral of the premeasure μ :*

$$h(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2}\, d\mu(\theta), \qquad z \in \mathbb{D};$$

to define this integral we integrate by parts.

Let f be a zero-free function in $A^{-\infty}$ such that $f(0) = 1$. According to Theorem 1.15, there is a premeasure $\mu_f \in B_\kappa^+$ such that

$$f(z) = \exp \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu_f(\theta).$$

1.3.3 Cyclicity

Every κ-bounded premeasure μ generates a finite Borel measure on Carleson sets, i.e. on closed subsets of $\mathbb{T}$ which have measure zero, and whose complementary arcs I_n on $\mathbb{T}$ satisfy the relation

$$\sum_n |I_n| \log \frac{2\pi e}{|I_n|} < \infty.$$

This Borel measure (defined on $\mathcal{B}$: the set of all Borel sets $B \subset \mathbb{T}$ such that $\overline{B}$ is Carleson set) is called the κ-singular part of κ-bounded premeasure μ and denoted by μ_s.

A description of all closed ideals (invariant subspaces for the operator of multiplication by z) of the topological algebra $A^{-\infty}$ is given in [32]. Each such zero-free closed ideal (zero-free z-invariant subspace) is uniquely characterized by its κ-singular measure which is completely analogous to the case of the invariant subspaces of H^2 described in the classical theory of Beurling [4].

Theorem 1.16 *[32] An element f in Korenblum space $A^{-\infty}$ is cyclic if and only if*

1. *$f(z)$ has no zeros in $\mathbb{D}$.*
2. *The κ-singular measure associated with f is zero.*

Theorem 1.17 *[24, Theorem 7.3] Let $0 < p < \infty$, and let $f \in A^q$, $q > p$. Then f is cyclic in A^p if and only if f has no zeros in $\mathbb{D}$ and the κ-singular measure associated with f is zero.*

Hopefully, new information on the structure of the closed ideals in $A^{-\infty}$ would mark at the same time some progress in the cyclicity problem for Bergman spaces A^p.

For other applications of the premeasures technique, one should mention a paper by N. G. Makarov in [37] describing z-invariant subspaces of the space C^∞ (the space of infinitely differentiable functions on the unit circle $\mathbb{T}$), and a paper by B. Korenblum [35] on the zero sets in A^p, A^{-p} spaces.

1.4 Results and presentation of the work

1.4.1 Cyclic vectors in Korenblum type spaces

In this section we extend the results of two papers by Korenblum [31, 32] on Λ-bounded premeasures (see also [24, Chapter 7]) from the case $\Lambda(t) = \log(1/t)$ to the general case.

In the following a majorant Λ will always denote a positive non-increasing convex differentiable function on (0,1] such that :

1. $\Lambda(0) = +\infty$
2. $t\Lambda(t)$ is a continuous, non-decreasing and concave function on $[0, 1]$, and $t\Lambda(t) \to 0$ as $t \to 0$.
3. There exists $\alpha \in (0, 1)$ such that $t^\alpha \Lambda(t)$ is non-decreasing.

4. For some $C > 0$, every $t \in (0, 1)$ we have

$$\Lambda(t^2) \leq C\Lambda(t). \tag{1.3}$$

Typical examples of majorants Λ are $\log^+ \log^+(1/x)$, $(\log(1/x))^p$, $p > 0$.

Denote by $\mathcal{A}_\Lambda^{-\infty}$ the Korenblum type space associated with the majorant Λ, defined by

$$\mathcal{A}_\Lambda^{-\infty} = \cup_{c>0} \mathcal{A}_\Lambda^{-c} = \bigcup_{c>0} \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \leq \exp(c\Lambda(1 - |z|)) \Big\}.$$

Note that the set of all polynomials is dense in the topological algebra (with respect to pointwise multiplication and the natural injective limit topology) $\mathcal{A}_\Lambda^{-\infty}$. Furthermore, M_z and the point evaluation functionals are bounded on $\mathcal{A}_\Lambda^{-\infty}$.

Definition 1.18 *A premeasure μ is said to be Λ-bounded, if there is a positive number C_μ such that*

$$\mu(I) \leq C_\mu |I| \Lambda(|I|) \tag{1.4}$$

for any interval I.

The minimal number C_μ is called the norm of μ and is denoted by $\|\mu\|_\Lambda^+$; the set of all real premeasures μ such that $\|\mu\|_\Lambda^+ < +\infty$ is denoted by B_Λ^+.

Given a closed non-empty subset F of the unit circle $\mathbb{T}$, we define its Λ-entropy as follows :

$$Entr_\Lambda(F) = \sum_n |I_n| \Lambda(|I_n|),$$

where $\{I_n\}_n$ are the component arcs of $\mathbb{T} \setminus F$, and $|I|$ denotes the normalized Lebesgue measure of I on $\mathbb{T}$. We set $Entr_\Lambda(\emptyset) = 0$.

We say that a closed set F is a Λ-Carleson set if F is non-empty, has Lebesgue measure zero (i.e $|F| = 0$), and $Entr_\Lambda(F) < +\infty$.

Denote by $\mathcal{C}_\Lambda$ the set of all Λ-Carleson sets and by $\mathcal{B}_\Lambda$ the set of all Borel sets $B \subset \mathbb{T}$ such that $\overline{B} \in \mathcal{C}_\Lambda$.

We can now introduce the notion of Λ-singular measures.

Definition 1.19 *A function $\sigma : \mathcal{B}_\Lambda \to \mathbb{R}$ is called a Λ-singular measure if*

1. *σ is a finite Borel measure on every set in $\mathcal{C}_\Lambda$ (i.e. $\sigma\big|F$ is a Borel measure on $\mathbb{T}$).*
2. *There is a constant $C > 0$ such that*

$$|\sigma(F)| \leq C Entr_\Lambda(F)$$

for all $F \in \mathcal{C}_\Lambda$.

Given a premeasure μ in B_Λ^+, its Λ-singular part is defined by :

$$\mu_s(F) = -\sum_n \mu(I_n), \tag{1.5}$$

where $F \in \mathcal{C}_\Lambda$ and $\{I_n\}_n$ is the collection of complementary intervals to F in $\mathbb{T}$. Using the argument in [31, Theorem 6] one can see that μ_s extends to a Λ-singular measure on $\mathcal{B}_\Lambda$.

Next we introduce the notion of Λ-absolutely continuous premeasure, which will give us a cyclicity criterion.

Definition 1.20 *A premeasure μ in B_Λ^+ is said to be Λ-absolutely continuous if there exists a sequence of Λ-bounded premeasures $(\mu_n)_n$ such that :*

1. $\sup_n \|\mu_n\|_\Lambda^+ < +\infty$.
2. $\sup_{I\in\mathcal{B}(\mathbb{T})} |(\mu + \mu_n)(I)| \to 0$ *as* $n \to +\infty$.

The proof of the next result follows an argument by Korenblum in [32].

Theorem 1.21 *Let μ be a premeasure in B_Λ^+. Then μ is Λ-absolutely continuous if and only if its Λ-singular part μ_s is zero.*

The only if part holds in a more general situation considered by Korenblum, [36, Corollary, p.544]. On the other hand, the if part does not hold for differences of Λ-bounded premeasures (premeasures of Λ-bounded variation), see [36, Remark, p.544].

Harmonic functions of restricted growth and premeasures.
Every bounded harmonic function can be represented via the Poisson integral of its boundary values. In the following theorem we show that a large class of real-valued harmonic functions in the unit disk $\mathbb{D}$ can be represented as the Poisson integrals of Λ-bounded premeasures.

The following theorem is stated by Korenblum in [36, Theorem 1, p. 543] without proof, in a more general situation.

Theorem 1.22 *Let h be a real-valued harmonic function on the unit disk such that $h(0) = 0$ and*

$$h(z) = O(\Lambda(1-|z|)), \qquad |z| \to 1,\ z \in \mathbb{D}.$$

Then the following statements hold.

1. *For every open arc I of the unit circle $\mathbb{T}$ the following limit exists :*

$$\mu(I) = \lim_{r\to1^-} \mu_r(I) = \lim_{r\to1^-} \int_I h(r\xi)\,|d\xi| < \infty.$$

2. *μ is a Λ-bounded premeasure.*
3. *The function h is the Poisson integral of the premeasure μ :*

$$h(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\mu(\theta), \qquad z \in \mathbb{D}.$$

Conversely, every Λ-bounded premeasure μ generates a harmonic function $h(z)$ in $\mathbb{D}$ (the Poisson integral of μ) such that

$$h(z) = O(\Lambda(1-|z|)), \qquad |z| \to 1,\ z \in \mathbb{D}, \tag{1.6}$$

by the formula

$$h(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|e^{i\theta}-z|^2} d\mu.$$

Let f be a zero-free function in $\mathcal{A}_\Lambda^{-\infty}$ such that $f(0)=1$. According to Theorem 1.22, there is a premeasure $\mu_f := \mu \in B_\Lambda^+$ such that

$$f(z) = f_\mu(z) := \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta). \tag{1.7}$$

The following result follows immediately from Theorem 1.21.

Theorem 1.23 *Let $f \in \mathcal{A}_\Lambda^{-\infty}$ be a zero-free function such that $f(0)=1$. If $(\mu_f)_s \equiv 0$, then f is cyclic in $\mathcal{A}_\Lambda^{-\infty}$.*

From now on, we deal with the statements converse to Theorem 1.23. We establish two results valid for different growth ranges of the majorant Λ. More precisely, we consider the following growth and regularity assumptions :

$$\text{for every } c>0, \text{ the function } x \mapsto \exp\big[c\Lambda(1/x)\big] \text{ is concave for large } x, \tag{C1}$$

$$\lim_{t\to 0} \frac{\Lambda(t)}{\log(1/t)} = \infty. \tag{C2}$$

Examples of majorants Λ satisfying condition (C1) include

$$(\log(1/x))^p, \quad 0<p<1, \quad \text{and} \quad \log(\log(1/x)), \quad x \to 0.$$

Examples of majorants Λ satisfying condition (C2) include

$$(\log(1/x))^p, \quad p>1.$$

Thus, we consider majorants which grow less rapidly than the Korenblum majorant ($\Lambda(x) = \log(1/x)$) in Case 1 or more rapidly than the Korenblum majorant in Case 2.

Theorem 1.24 *Let $\mu \in B_\Lambda^+$, and let the majorant Λ satisfy condition* (C1)*. Then the function f_μ is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if $\mu_s \equiv 0$.*

Theorem 1.25 *Let $\mu \in B_\Lambda^+$, and let the majorant Λ satisfy condition* (C2)*. Then the function f_μ is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if $\mu_s \equiv 0$.*

Theorems 1.24 and 1.25 together give a positive answer to a conjecture by Deninger [14, Conjecture 42].

Now we give two examples that show how the cyclicity property of a fixed function changes in a scale of $\mathcal{A}_\Lambda^{-\infty}$ spaces.

Example 1.26 *Let* $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, *and let* $0 < \alpha_0 < 1$. *There exists a singular inner function* S_μ *such that*

$$S_\mu \textit{ is cyclic in } \mathcal{A}_{\Lambda_\alpha}^{-\infty} \iff \alpha > \alpha_0.$$

Example 1.27 *Let* $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, *and let* $0 < \alpha_0 < 1$. *There exists a premeasure* μ *such that* μ_s *is infinite,*

$$f_\mu \textit{ is cyclic in } \mathcal{A}_{\Lambda_\alpha}^{-\infty} \iff \alpha > \alpha_0,$$

where f_μ *is defined by* (1.7).

It looks like the subspaces $[f_\mu]_{\mathcal{A}_{\Lambda_\alpha}^{-\infty}}$, $\alpha \leq \alpha_0$, contain no nonzero Nevanlinna class functions. For a detailed discussion on Nevanlinna class generated invariant subspaces in the Bergman space (and in the Korenblum space) see [23, Chapter 6].

1.4.2 Cyclicity in weighted Bergman type spaces

In this part, we investigate the question of weak invertibility (cyclicity) of the atomic singular inner function $S(z) = e^{-\frac{1+z}{1-z}}$ with its measure μ concentrated at one point, in the weighted Bergman type spaces.

Given a positive non-increasing continuous function Λ on $(0, 1]$ and $E \subset \mathbb{T} = \partial\mathbb{D}$, we denote by $\mathcal{B}_{\Lambda,E}^\infty$ the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\Lambda,E,\infty} = \sup_{z\in\mathbb{D}} |f(z)|e^{-\Lambda(\operatorname{dist}(z,E))} < +\infty,$$

and by $\mathcal{B}_{\Lambda,E}^{\infty,0}$ its separable subspace

$$\mathcal{B}_{\Lambda,E}^{\infty,0} = \big\{f \in \mathcal{B}_{\Lambda,E}^\infty : \lim_{\operatorname{dist}(z,E)\to 0} |f(z)|e^{-\Lambda(\operatorname{dist}(z,E))} = 0\big\}.$$

Analogously, integrating with respect to area measure on the disc, we define the spaces $\mathcal{B}_{\Lambda,E}^p$, $1 \leq p < \infty$:

$$\mathcal{B}_{\Lambda,E}^p = \Big\{f \in \operatorname{Hol}(\mathbb{D}) : \|f\|_{\Lambda,E,p}^p = \int_{\mathbb{D}} |f(z)|^p e^{-p\Lambda(\operatorname{dist}(z,E))} < +\infty\Big\}.$$

If $E = \mathbb{T}$, we use the notations $\mathcal{B}_\Lambda^\infty$, $\mathcal{B}_\Lambda^{\infty,0}$, $\mathcal{B}_\Lambda^p$. Let us remark that either $\Lambda(0^+) = +\infty$ or $\mathcal{B}_\Lambda^\infty = H^\infty$, $\mathcal{B}_\Lambda^{\infty,0} = \{0\}$, $\mathcal{B}_\Lambda^p = \mathcal{B}_0^p$.

Given a sequence of positive numbers $w = (w_n)$ such that $|\log w_n| = o(n)$, $n \to \infty$, we define by H^2_w the space of functions f analytic in the unit disc and such that

$$f(z) = \sum_{n\geq 0} a_n z^n, \qquad \sum_{n\geq 0} \frac{|a_n|^2}{w_n} < \infty.$$

It is known that for log-convex sequences w such spaces coincide with the $\mathcal{B}^2_{\Lambda,\mathbb{T}}$ spaces for Λ defined by w (see [8]).

The cyclicity question in the weighted Bergman type spaces $\mathcal{B}^p_\Lambda$, $1 \leq p < \infty$, goes back to Carleman and Keldysh. In particular, Keldysh [25] proved in 1945 that the singular inner function with one point singular mass $S(z) = e^{-\frac{1+z}{1-z}}$ is not cyclic in $\mathcal{B}^2_0$.

In 1964 Beurling [5] studied the cyclicity of the function S in space $\bigcup_{k\geq 1} H^2_{w^k}$, equipped with the natural topology of an inductive limit (this space is a topological algebra with respect to ordinary multiplication of functions). Using the Bernstein's approximation theorem he produced under mild restrictions on the regularity of growth, a necessary and sufficient condition for the localization of all principal ideals (in other words, a necessary and sufficient condition for each function in $\bigcup_{k\geq 1} H^2_{w^k}$ lacking zeros in $\mathbb{D}$ to be cyclic in $\bigcup_{k\geq 1} H^2_{w^k}$). This condition is the divergence of the series

$$\sum_{n\geq 1} \frac{\log w_n}{n^{3/2}}.$$

In 1974 Nikolski [39, Section 2.6] proved that if $\liminf_{t\to 0} \frac{\Lambda(t)}{\log 1/t} > 0$, and S is cyclic in $\mathcal{B}^\infty_\Lambda$, then

$$\int_0 \sqrt{\frac{\Lambda(t)}{t}} dt = \infty. \tag{1.8}$$

In the opposite direction, he proved that if

$$\text{the function } t \mapsto t\Lambda'(t) \text{ does not decrease,} \tag{1.9}$$

and (1.8) holds, then S is cyclic in $\mathcal{B}^\infty_\Lambda$. Since the proof relied on the quasianalyticity property of an auxiliary class of functions, this convexity type condition (1.9) was indispensable here.

In the first part of this subsection we prove the following results :

Theorem 1.28 *Let Λ be a positive non-increasing continuous function on $(0, 1]$. Then $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^{\infty,0}_\Lambda$ if and only if Λ satisfies (1.8).*

Theorem 1.29 *Let $1 \leq p < \infty$ and let Λ be a positive non-increasing continuous function on $(0, 1]$. Then $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^p_\Lambda$ if and only if Λ satisfies (1.8).*

Thus, we are able to get rid of any regularity condition on Λ. That is possible because we use the so-called resolvent transform method exposed, for example, in [15, 8]. This technique was introduced by Carleman and Gelfand; it was later rediscovered and used upon by Domar to study closed ideals in Banach algebras.

In 1986 Gevorkyan and Shamoyan [20] obtained a necessary and sufficient condition,

$$\int_0 \Lambda(t)\,dt = \infty, \tag{1.10}$$

for cyclicity of S in $\mathcal{B}^{\infty,0}_{\Lambda,\{1\}}$ under some regularity conditions on Λ. Recently, El-Fallah, Kellay, and Seip [17] improved the results of Beurling and Nikolski for cyclicity of bounded zero-free functions in H^2_w spaces. Furthermore, improving on the result by Gevorkyan–Shamoyan they obtained that (1.10) is necessary and sufficient for cyclicity of S in $\mathcal{B}^{\infty,0}_{\Lambda,\{1\}}$, the only regularity condition being that Λ is decreasing. Their method of proof (applying the Corona theorem) is quite different from what we use here.

It is now a natural question to describe Λ such that S is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$, $\mathcal{B}^{2}_{\Lambda,E}$ in terms of the behavior of E near the point 1. Our method applies directly in the case where E is a closed arc containing 1. Furthermore, for sufficiently regular Λ we get necessary and sufficient conditions in terms of E.

In the second part of this subsection, for sufficiently regular Λ we get necessary and sufficient conditions in terms of E. More precisely, for Λ defined by $\Lambda(t) = \frac{1}{tw(t)^2}$ with sufficiently regular w, the function S is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$ if and only if one of the following three quantities is infinite :

$$\int_{e^{it}\in E} \frac{dt}{|t|w(|t|)}, \quad \int_0 \frac{dt}{|t|w^2(|t|)}, \quad \sum_n \frac{1}{w(b_n)^2} \log\Big[1 + \Big(1 - \frac{a_n}{b_n}\Big) w(b_n)\Big], \tag{1.11}$$

where the sums runs by all the complementary arcs to E : (e^{ia_n}, e^{ib_n}), (e^{-ib_n}, e^{-ia_n}), $0 < a_n < b_n$.

Two simple examples of the set E are considered in the following corollary

Corollary 1.30 *Let Λ be the function described above. If $E = \{\exp(i \cdot 2^{-n})\}_{n\geq 1} \cup \{1\}$, then S is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$ if and only if*

$$\sum_{n\geq 1} \frac{\log w(2^{-n})}{w(2^{-n})^2} = +\infty\,;$$

if $E = \{\exp(i \cdot 2^{-2^n})\}_{n\geq 1} \cup \{1\}$, then S is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$ if and only if

$$\int_0 \frac{dt}{tw(t)^2} = +\infty,$$

and we return to the Gevorkyan–Shamoyan condition valid for $E = \{1\}$.

Next we give two more applications of the general criterion (1.11).

First, we get a result interpolating between the theorems of Nikolski and Gevorkyan–Shamoyan. Let us introduce a condition

$$\int_0 \frac{\Lambda(t)^{1-\beta}}{t^\beta}\,dt = +\infty, \qquad 0 \le \beta \le \frac{1}{2} \tag{C_β}$$

(compare to those by Nikolski ($\int_0 \sqrt{\Lambda(t)/t}\,dt = +\infty$, $\beta = 1/2$) and by Gevorkyan–Shamoyan ($\int_0 \Lambda(t)\,dt = +\infty$, $\beta = 0$)).

For

$$\Lambda_\alpha(t) = \frac{1}{t \log^\alpha(1/t)}$$

we have

$$\Lambda_\alpha \in (C_\beta) \iff \alpha(1-\beta) \le 1.$$

Theorem 1.31 *Let* $0 \le \beta \le 1/2$, $a_n = \exp(-n^{1-\beta})$, $n \ge 1$, $E_\beta = \{e^{ia_n}\}_{n\ge1} \cup \{1\}$. *The function* $S(z) = e^{-\frac{1+z}{1-z}}$ *is cyclic in* $\mathcal{B}^{\infty,0}_{\Lambda_\alpha,E_\beta}$ *if and only if* $\Lambda_\alpha \in (C_\beta)$.

We also give another application of our result, in the case when E is the Cantor ternary set. Denote by κ the Hausdorff dimension of E (see [18, Section 1.5]), $\kappa = \frac{\log 2}{\log 3}$.

Theorem 1.32 *Let E be the Cantor ternary set. The function* $S(z) = e^{-\frac{1+z}{1-z}}$ *is cyclic in* $\mathcal{B}^{\infty,0}_{\Lambda_\alpha,E}$ *if and only if*

$$\alpha \le \frac{1}{1-\frac{\kappa}{2}}.$$

Chapitre 2

Cyclic vectors in Korenblum type spaces

In this chapter, a majorant Λ will always denote a positive non-increasing convex differentiable function on (0,1] such that :

1. $\Lambda(0) = +\infty$
2. $t\Lambda(t)$ is a continuous, non-decreasing and concave function on $[0,1]$, and $t\Lambda(t) \to 0$ as $t \to 0$.
3. There exists $\alpha \in (0,1)$ such that $t^\alpha \Lambda(t)$ is non-decreasing.
4. There exists $C > 0$, such that
$$\Lambda(t^2) \leq C\Lambda(t). \tag{2.1}$$

Typical examples of majorants Λ are $\log^+ \log^+(1/x)$, $(\log(1/x))^p$, $p > 0$.

Furthermore, we shall be interested mainly in studying cyclic vectors in the space $\mathcal{A}_\Lambda^{-\infty}$, by generalized the theory of premeasures introduced by Korenblum ; here $\mathcal{A}_\Lambda^{-\infty}$ is the Korenblum type space associated with the majorant Λ, defined by

$$\mathcal{A}_\Lambda^{-\infty} = \cup_{c>0}\mathcal{A}_\Lambda^{-c} = \bigcup_{c>0} \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : |f(z)| \leq \exp(c\Lambda(1-r)) \Big\}.$$

2.1 Λ-bounded premeasures

In this section we extend the results of two papers by Korenblum [31, 32] on Λ-bounded premeasures (see also [24, Chapter 7]) from the case $\Lambda(t) = \log(1/t)$ to the general case.

Let $\mathcal{B}(\mathbb{T})$ be the set of all (open, half-open and closed) arcs of $\mathbb{T}$ including all the single points and the empty set. The elements of $\mathcal{B}(\mathbb{T})$ will be called intervals.

Definition 2.1 *A real function defined on $\mathcal{B}(\mathbb{T})$ is called a premeasure if the following conditions hold :*

1. $\mu(\mathbb{T}) = 0$
2. $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$ *for every* $I_1, I_2 \in \mathcal{B}(\mathbb{T})$ *such that* $I_1 \cap I_2 = \emptyset$ *and* $I_1 \cup I_2 \in \mathcal{B}(\mathbb{T})$

3. $\lim_{n\to+\infty} \mu(I_n) = 0$ *for every sequence of embedded intervals,* $I_{n+1} \subset I_n$*,* $n \geq 1$*, such that* $\bigcap_n I_n = \emptyset$.

Given a premeasure μ, we introduce a real valued function $\overset{\curlywedge}{\mu}$ on $(0, 2\pi]$ defined as follows :

$$\overset{\curlywedge}{\mu}(\theta) = \mu(I_\theta),$$

where

$$I_\theta = \big\{\xi \in \mathbb{T} : 0 \leq \arg \xi < \theta\big\}.$$

The function $\overset{\curlywedge}{\mu}$ satisfies the following properties :

(a) $\overset{\curlywedge}{\mu}(\theta^-)$ exists for every $\theta \in (0{,}2\pi]$ and $\overset{\curlywedge}{\mu}(\theta^+)$ exists for every $\theta \in [0, 2\pi)$

(b) $\overset{\curlywedge}{\mu}(\theta) = \lim_{t\to\theta^-} \overset{\curlywedge}{\mu}(t)$ for all $\theta \in (0{,}2\pi]$

(c) $\overset{\curlywedge}{\mu}(2\pi) = \lim_{\theta\to 0^+} \overset{\curlywedge}{\mu}(\theta) = 0$.

Furthermore, the function $\overset{\curlywedge}{\mu}(\theta)$ has at most countably many points of discontinuity.

Definition 2.2 *A real premeasure* μ *is said to be* Λ*-bounded, if there is a positive number* C_μ *such that*

$$\mu(I) \leq C_\mu |I| \Lambda(|I|) \tag{2.2}$$

for any interval I.

The minimal number C_μ is called the norm of μ and is denoted by $\|\mu\|_\Lambda^+$; the set of all real premeasures μ such that $\|\mu\|_\Lambda^+ < +\infty$ is denoted by B_Λ^+.

Definition 2.3 *A sequence of premeasures* $\{\mu_n\}_n$ *is said to be* Λ*-weakly convergent to a premeasure* μ *if :*

1. $\sup_n \|\mu_n\|_\Lambda^+ < +\infty$*, and*
2. *for every point* θ *of continuity of* $\overset{\curlywedge}{\mu}$ *we have* $\lim_{n\to\infty} \overset{\curlywedge}{\mu}_n(\theta) = \overset{\curlywedge}{\mu}(\theta)$.

In this situation, the limit premeasure μ is Λ-bounded.

Given a closed non-empty subset F of the unit circle $\mathbb{T}$, we define its Λ-entropy as follows :

$$Entr_\Lambda(F) = \sum_n |I_n| \Lambda(|I_n|),$$

where $\{I_n\}_n$ are the component arcs of $\mathbb{T} \setminus F$, and $|I|$ denotes the normalized Lebesgue measure of I on $\mathbb{T}$. We set $Entr_\Lambda(\emptyset) = 0$.

We say that a closed set F is a Λ-Carleson set if F is non-empty, has Lebesgue measure zero (i.e $|F| = 0$), and $Entr_\Lambda(F) < +\infty$.

Denote by $\mathcal{C}_\Lambda$ the set of all Λ-Carleson sets and by $\mathcal{B}_\Lambda$ the set of all Borel sets $B \subset \mathbb{T}$ such that $\overline{B} \in \mathcal{C}_\Lambda$.

Definition 2.4 *A function $\sigma : \mathcal{B}_\Lambda \to \mathbb{R}$ is called a Λ-singular measure if*

1. *σ is a finite Borel measure on every set in $\mathcal{C}_\Lambda$ (i.e. $\sigma\big|F$ is a Borel measure on $\mathbb{T}$).*
2. *There is a constant $C > 0$ such that*

$$|\sigma(F)| \le C Entr_\Lambda(F)$$

for all $F \in \mathcal{C}_\Lambda$.

Given a premeasure μ in B^+_Λ, its Λ-singular part is defined by :

$$\mu_s(F) = -\sum_n \mu(I_n), \tag{2.3}$$

where $F \in \mathcal{C}_\Lambda$ and $\{I_n\}_n$ is the collection of complementary intervals to F in $\mathbb{T}$. Using the argument in [31, Theorem 6] one can see that μ_s extends to a Λ-singular measure on $\mathcal{B}_\Lambda$.

Proposition 2.5 *If μ is a Λ-bounded premeasure, $F \in \mathcal{C}_\Lambda$, then $\mu_s\big|F$ is finite and non-positive.*

Proof. Let $F \in \mathcal{C}_\Lambda$. We are to prove that $\mu_s(F) \le 0$.

Let $\{I_n\}_n$ be the (possibly finite) sequence of the intervals complementary to F in $\mathbb{T}$. For $N \ge 1$, we consider the disjoint intervals $\{J^N_n\}_{1\le n\le N}$ such that $\mathbb{T} \setminus \bigcup_{n=1}^N I_n = \bigcup_n^N J^N_n$. Then

$$-\sum_{n=1}^N \mu(I_n) = \sum_{n=1}^N \mu(J^N_n) \le \|\mu\|^+_\Lambda \sum_{n=1}^N |J^N_n|\Lambda(|J^N_n|).$$

Furthermore, each interval J^N_n is covered by intervals $I_m \subset J^N_n$ up to a set of measure zero, and $\max_{1\le n\le N} |J^N_n| \to 0$ as $N \to \infty$ (If the sequence $\{I_n\}_n$ is finite, then all J^N_n are single points for the corresponding N). Therefore,

$$-\sum_{n=1}^N \mu(I_n) \le \|\mu\|^+_\Lambda \sum_{n=1}^N \sum_{I_m\subset J^N_n} |I_m|\Lambda(|I_m|) \le \|\mu\|^+_\Lambda \sum_{n>N} |I_n|\Lambda(|I_n|).$$

Since F is a Λ-Carleson set,

$$-\lim_{N\to\infty} \sum_{n=1}^N \mu(I_n) \le 0.$$

Thus, $\mu_s\big|F \le 0$. □

Given a closed subset F of $\mathbb{T}$, we denote by F^δ its δ-neighborhood :

$$F^\delta = \{\zeta \in \mathbb{T} : d(\zeta, F) \le \delta\}.$$

Proposition 2.6 *Let μ be a Λ-bounded premeasure and let μ_s be its Λ-singular part. Then for every $F \in \mathcal{C}_\Lambda$ we have*

$$\mu_s(F) = \lim_{\delta\to 0} \mu(F^\delta). \tag{2.4}$$

Proof. Let $F \in \mathcal{C}_\Lambda$, and let $\{I_n\}_n$, $|I_1| \geq |I_2| \geq \dots$, be the intervals of the complement to F in $\mathbb{T}$. We set

$$I_n^{(\delta)} = \big\{e^{i\theta} : \operatorname{dist}(e^{i\theta}, \mathbb{T} \setminus I_n) > \delta\big\}.$$

Then for $|I_n| \geq 2\delta$, we have

$$I_n = I_n^1 \sqcup I_n^{(\delta)} \sqcup I_n^2$$

with $|I_n^1| = |I_n^2| = \delta$. We see that

$$\mu(F^\delta) = -\sum_{|I_n|>2\delta} \mu(I_n^{(\delta)}).$$

Using relation (2.3) we obtain that

$$\begin{aligned} -\mu_s(F) &= \sum_n \mu(I_n) \\ &= \sum_{|I_n|\leq 2\delta} \mu(I_n) + \sum_{|I_n|>2\delta} \Big[\mu(I_n^1) + \mu(I_n^{(\delta)}) + \mu(I_n^2)\Big] \\ &= \sum_{|I_n|\leq 2\delta} \mu(I_n) - \mu(F^\delta) + \sum_{|I_n|>2\delta} \big[\mu(I_n^1) + \mu(I_n^2)\big]. \end{aligned}$$

Therefore,

$$\mu(F^\delta) - \mu_s(F) = \sum_{|I_n|\leq 2\delta} \mu(I_n) + \sum_{|I_n|>2\delta} \big[\mu(I_n^1) + \mu(I_n^2)\big]$$

The first sum tends to zero as $\delta \to 0$, and it remains to prove that

$$\lim_{\delta\to 0} \sum_{|I_n|>2\delta} \mu(I_n^1) = 0. \tag{2.5}$$

We have

$$\sum_{|I_n|>2\delta} \mu(I_n^1) \leq C \sum_{|I_n|>\delta} \delta\Lambda(\delta) = C \sum_{|I_n|>\delta} \frac{\delta\Lambda(\delta)}{|I_n|\Lambda(|I_n|)} \cdot |I_n|\Lambda(|I_n|).$$

Since the function $t \mapsto t\Lambda(t)$ does not decrease, we have

$$\frac{\delta\Lambda(\delta)}{|I_n|\Lambda(|I_n|)} \leq 1, \qquad |I_n| > \delta.$$

Furthermore,

$$\lim_{\delta\to 0} \frac{\delta\Lambda(\delta)}{|I_n|\Lambda(|I_n|)} = 0, \qquad n \geq 1.$$

Since

$$\sum_{n\geq 1} |I_n|\Lambda(|I_n|) < \infty,$$

we conclude that (2.5), and, hence, (2.4) hold.

□

Definition 2.7 *A premeasure μ in B_Λ^+ is said to be Λ-absolutely continuous if there exists a sequence of Λ-bounded premeasures $(\mu_n)_n$ such that :*

1. $\sup_n \|\mu_n\|_\Lambda^+ < +\infty$.
2. $\sup_{I\in\mathcal{B}(\mathbb{T})} |(\mu+\mu_n)(I)| \to 0$ *as* $n \to +\infty$.

Theorem 2.8 *Let μ be a premeasure in B_Λ^+. Then μ is Λ-absolutely continuous if and only if its Λ-singular part μ_s is zero.*

The only if part holds in a more general situation considered by Korenblum, [36, Corollary, p.544]. On the other hand, the if part does not hold for differences of Λ-bounded premeasures (premeasures of Λ-bounded variation), see [36, Remark, p.544].

To prove this theorem we need several lemmas. The first one is a linear programming lemma from [24, Chapter 7].

Lemma 2.9 *Consider the following system of $N(N+1)/2$ linear inequalities in N variables $x_1, \dots, x_N$*

$$\sum_{j=k}^{l} x_j \leq b_{k,l}, \qquad 1 \leq k \leq l \leq N,$$

subject to the constraint : $x_1 + x_2 + \cdots + x_N = 0$. This system has a solution if and only if

$$\sum_n b_{k_n,l_n} \geq 0$$

for every simple covering $\mathcal{P} = \{[k_n, l_n]\}_n$ of $[1, N]$.

The following lemma gives a necessary and sufficient conditions for a premeasure in B_Λ^+ to be Λ-absolutely continuous.

Lemma 2.10 *Let μ be a Λ-bounded premeasure. Then μ is Λ-absolutely continuous if and only if there is a positive constant $C > 0$ such that for every $\varepsilon > 0$ there exists a positive M such that the system*

$$\left\{ \begin{array}{rcl} x_{k,l} & \leq & M|I_{k,l}|\Lambda(|I_{k,l}|) \\ \mu(I_{k,l}) + x_{k,l} & \leq & \min\{C|I_{k,l}|\Lambda(|I_{k,l}|), \varepsilon\} \\ x_{k,l} & = & \sum_{s=k}^{l-1} x_{s,s+1} \\ x_{0,N} & = & 0 \end{array} \right. \tag{2.6}$$

in variables $x_{k,l}$, $0 \le k < l \le N$, has a solution for every positive integer N. Here $I_{k,l}$ are the half-open arcs of $\mathbb{T}$ defined by

$$I_{k,l} = \Big\{ e^{i\theta} : 2\pi \frac{k}{N} \le \theta < 2\pi \frac{l}{N} \Big\}.$$

Proof.

Suppose that μ is Λ-absolutely continuous and denote by $\{\mu_n\}$ a sequence of Λ-bounded premeasures satisfying the conditions of Definition 2.7. Set

$$C = \sup_n \|\mu + \mu_n\|_{\Lambda}^{+}, \qquad M = \sup_n \|\mu_n\|_{\Lambda}^{+},$$

and let $\varepsilon > 0$. For large n, the numbers $x_{k,l} = \mu_n(I_{k,l})$, $0 \le k < l \le N$, satisfy relations (2.6) for all N.

Conversely, suppose that for some $C > 0$ and for every $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ such that for every N there are $\{x_{k,l}\}_{k,l}$ (depending on N) satisfying relations (2.6). We consider the measures $d\mu_N$ defined on $I_{s,s+1}$, $0 \le s < N$, by

$$d\mu_N(\xi) = \frac{x_{s,s+1}}{|I_{s,s+1}|} |d\xi|,$$

where $|d\xi|$ is normalized Lebesgue measure on the unit circle $\mathbb{T}$. To show that $\mu_N \in B_{\Lambda}^{+}$, it suffices to verify that the quantity $\sup_I \frac{\mu(I)}{|I|\Lambda(|I|)}$ is finite for every interval $I \in \mathcal{B}(\mathbb{T})$. Fix $I \in \mathcal{B}(\mathbb{T})$ such that $1 \notin I$.

If $I \subset I_{k,k+1}$, then

$$\mu_N(I) = \frac{x_{k,k+1}}{|I_{k,k+1}|} |I| \le \frac{x_{k,k+1}}{|I_{k,k+1}| \Lambda(|I_{k,k+1}|)} |I| \Lambda(|I|) \le M |I| \Lambda(|I|).$$

If $I = I_{k,l}$, then

$$\mu_N(I_{k,l}) = \sum_{s=k}^{l-1} \mu_N(I_{s,s+1}) = \sum_{s=k}^{l-1} x_{s,s+1} = x_{k,l} \le M |I_{k,l}| \Lambda(|I_{k,l}|).$$

Otherwise, denote by $I_{k,l}$ the largest interval such that $I_{k,l} \subset I$. We have

$$\begin{aligned} \mu_N(I) &= \mu_N(I_{k,l}) + \mu_N(I \setminus I_{k,l}) \\ &\le M|I_{k,l}|\Lambda(|I_{k,l}|) + M|I_{k-1,k}|\Lambda(|I_{k-1,k}|) + M|I_{l,l+1}|\Lambda(|I_{l,l+1}|) \\ &\le 3M|I|\Lambda(|I|). \end{aligned}$$

Thus, μ_N is a Λ-bounded premeasure. Next, using a Helly-type selection theorem for premeasures due to Cyphert and Kelingos [13, Theorem 2], we can find a Λ-bounded premeasure ν and a subsequence $\mu_{N_k} \in B_{\Lambda}^{+}$ such that $\{\mu_{N_k}\}_k$ converge Λ-weakly to ν. Furthermore, ν satisfies the following conditions :

$\nu(J) \leq 3M|J|\Lambda(|J|)$ and $\mu(J)+\nu(J) \leq \min\{C|J|\Lambda(|J|),\varepsilon\}$ for every interval $J \subset \mathbb{T}\setminus\{1\}$.

Now, if I is an interval containing the point 1, we can represent it as $I = I_1 \sqcup \{1\} \sqcup I_2$, for some (possibly empty) intervals I_1 and I_2. Then

$$\begin{aligned} \mu(I)+\nu(I) &= (\mu+\nu)(I_1)+(\mu+\nu)(I_2)+(\mu+\nu)(\{1\}) \\ &\leq (\mu+\nu)(I_1)+(\mu+\nu)(I_2). \end{aligned}$$

Therefore, for every $I \in \mathcal{B}(\mathbb{T})$ we have $\mu(J)+\nu(J) \leq 2\varepsilon$. Since $(\mu+\nu)(\mathbb{T}\setminus I) = -\mu(I)-\nu(I)$, we have

$$|\mu(J)+\nu(J)| \leq 2\varepsilon.$$

Thus μ is Λ-absolutely continuous.

□

Lemma 2.11 *Let $\mu \in B_\Lambda^+$ be not Λ-absolutely continuous. Then for every $C>0$ there is $\varepsilon>0$ such that for all $M>0$, there exists a simple covering of $\mathbb{T}$ by a finite number of half-open intervals $\{I_n\}_n$, satisfying the relation*

$$\sum_n \min\{\mu(I_n)+M|I_n|\Lambda(|I_n|), C|I_n|\Lambda(|I_n|),\varepsilon\} < 0.$$

Proof. By Lemma 2.10, for every $C>0$ there exists a number $\varepsilon>0$ such that for all $M>0$, the system (2.6) has no solutions for some $N \in \mathbb{N}$. In other words, there are no $\{x_{k,l}\}_{k,l}$ such that :

$$\sum_{s=k}^{l-1}\mu(I_{s,s+1})+x_{s,s+1} \leq \min\left\{\mu(I_{k,l})+M|I_{k,l}|\Lambda(|I_{k,l}|), C|I_{k,l}|\Lambda(|I_{k,l}|),\varepsilon\right\} \qquad (2.7)$$

with $x_{k,l} = \sum\limits_{s=k}^{l-1} x_{s,s+1}$ and $x_{0,N}=0$.

We set $X_j = \mu(I_{j,j+1})+x_{j,j+1}$, and

$$b_{k,l} = \min\left\{\mu(I_{k,l+1})+M|I_{k,l+1}|\Lambda(|I_{k,l+1}|), C|I_{k,l+1}|\Lambda(|I_{k,l+1}|),\varepsilon\right\}.$$

Then relations (2.7) are rewritten as

$$\sum_{j=k}^{l} X_j \leq b_{k,l}, \qquad 0 \leq k < l \leq N-1.$$

Therefore, we are in the conditions of Lemma 2.9 with variables X_j. We conclude that there is a simple covering of the circle $\mathbb{T}$ by a finite number of half-open intervals $\{I_n\}$ such that

$$\sum_n \min\left\{\mu(I_n)+M|I_n|\Lambda(|I_n|), C|I_n|\Lambda(|I_n|),\varepsilon\right\} < 0.$$

□

In the following lemma we give a normal families type result for the Λ-Carleson sets.

Lemma 2.12 *Let $\{F_n\}_n$ be a sequence of sets on the unit circle, and let each F_n be a finite union of closed intervals. We assume that*

(i) $|F_n| \to 0, \qquad n \to \infty,$

(ii) $Entr_\Lambda(F_n) = O(1), \qquad n \to \infty.$

Then there exists a subsequence $\{F_{n_k}\}_k$ and a Λ-Carleson set F such that :
for every $\delta > 0$ there is a natural number N with

(a) $F_{n_k} \subset F^\delta$

(b) $F \subset F^\delta_{n_k}$.

for all $k \geq N$.

Proof. Let $\{I_{k,n}\}_k$ be the complementary arcs to F_n such that $|I_{1,n}| \geq |I_{2,n}| \geq \ldots$. We show first that the sequence $\{|I_{1,n}|\}_n$ is bounded away from zero. Since the function Λ is non-increasing, we have

$$Entr_\Lambda(F_n) = \sum_k |I_{k,n}|\Lambda(|I_{k,n}|) \geq |\mathbb{T} \setminus F_n|\Lambda(|I_{1,n}|),$$

and therefore,

$$\frac{Entr_\Lambda(F_n)}{|\mathbb{T} \setminus F_n|} \geq \Lambda(|I_{1,n}|).$$

Now the conditions (i) and (ii) of lemma and the fact that $\Lambda(0^+) = +\infty$ imply that the sequence $\{|I_{1,n}|\}_n$ is bounded away from zero.

Given a subsequence $\{F_k^{(m)}\}_k$ of F_n, we denote by $(I_{j,k}^{(m)})_j$ the complementary arcs to $F_k^{(m)}$. Let us choose a subsequence $\{F_k^{(1)}\}_k$ such that

$$I_{1,k}^{(1)} = (a_k^{(1)}, b_k^{(1)}) \to (a^1, b^1) = J_1$$

as $k \to +\infty$, where J_1 is a non-empty open arc.

If $|J_1| = 1$, then $F = \mathbb{T} \setminus J_1$ is a Λ-Carleson set, and we are done : we can take $\{F_{n_k}\}_k = \{F_k^{(1)}\}_k$.

Otherwise, if $|J_1| < 1$, then, using the above method we show that

$$\Lambda(|I_{2,k}^{(1)}|) \leq \frac{Entr_\Lambda(F_k^{(1)})}{|\mathbb{T} \setminus F_k^{(1)}| - |I_{1,k}^{(1)}|}.$$

Since $\lim_{k\to+\infty} |\mathbb{T} \setminus F_k^{(1)}| - |I_{1,k}^{(1)}| = 1 - |J_1| > 0$, the sequence $\Lambda(|I_{2,k}^{(1)}|)$ is bounded, and hence, the sequence $|I_{2,k}^{(1)}|$ is bounded away from zero. Next we choose a subsequence

$\{F_k^{(2)}\}_k$ of $\{F_k^{(1)}\}_k$ such that the arcs $I_{2,k}^{(2)} = (a_k^2, b_k^2)$ tend to $(a^{(2)}, b^{(2)}) = J_2$, where J_2 is a non-empty open arc. Repeating this process we can have two possibilities. First, suppose that after a finite number of steps we have $|J_1| + \ldots + |J_m| = 1$, and then we can take $\{F_{n_k}\}_k = \{F_k^{(m)}\}_k$,

$$I_{j,k}^{(m)} \to J_j, \qquad 1 \le j \le m,$$

as $k \to +\infty$, and $F = \mathbb{T} \setminus \bigcup_{j=1}^{m} J_j$ is Λ-Carleson.

Now, if the number of steps is infinite, then using the estimate

$$\Lambda(|J_l|) \le \frac{\sup_n \{Entr_\Lambda(F_n)\}}{1 - \sum_{k=1}^{l-1} |J_k|},$$

and the fact $|J_m| \to 0$ as $m \to \infty$, we conclude that

$$\sum_{j=1}^{\infty} |J_j| = 1.$$

We can set $\{F_{n_k}\}_k = \{F_m^{(m)}\}_m$, $F = \mathbb{T} \setminus \bigcup_{j\ge 1} J_j$.

In all three situations the properties (a) and (b) follow automatically. □

Proof of Theorem 2.8

First we suppose that μ is Λ-absolutely continuous, and prove that $\mu_s = 0$. Choose a sequence μ_n of Λ-bounded premeasures satisfying the properties (1) and (2) of Definition 2.7. Let F be a Λ-Carleson set and let $(I_n)_n$ be the sequence of the complementary arcs to F. Denote by $(\mu + \mu_n)_s$ the Λ-singular part of $\mu + \mu_n$. Then

$$\begin{aligned} -(\mu+\mu_n)_s(F) &= \sum_k (\mu+\mu_n)(I_k) \\ &= \sum_{k\le N} (\mu+\mu_n)(I_k) + \sum_{k>N} (\mu+\mu_n)(I_k) \\ &\le \sum_{k\le N} (\mu+\mu_n)(I_k) + C \sum_{k>N} |I_k|\Lambda(|I_k|) \end{aligned}$$

Using the property (2) of Definition 2.7 we obtain that

$$-\liminf_{n\to\infty} (\mu+\mu_n)_s(F) \le C \sum_{k>N} |I_k|\Lambda(|I_k|).$$

Since $F \in \mathcal{C}_\Lambda$, we have $\sum_{k>N} |I_k|\Lambda(|I_k|) \to 0$ as $N \to +\infty$, and hence $\liminf_{n\to\infty}(\mu + \mu_n)_s(F) \ge 0$. Since $(\mu + \mu_n) \in B_\Lambda^+$, by Proposition 2.5 its Λ-singular part is non-positive. Thus $\lim_{n\to\infty}(\mu + \mu_n)_s(F) = 0$ for all $F \in \mathcal{C}_\Lambda$, which proves that $\mu_s = 0$.

Now, let us suppose that μ is not Λ-absolutely continuous. We apply Lemma 2.11 with $C = 4\|\mu\|^+_\Lambda$ and find $\varepsilon > 0$ such that for all $M > 0$, there is a simple covering of circle $\mathbb{T}$ by a half-open intervals $\{I_1, I_2, \dots I_N\}$ such that

$$\sum_n \min\left\{\mu(I_n) + M|I_n|\Lambda(|I_n|), 4\|\mu\|^+_\Lambda |I_n|\Lambda(|I_n|), \varepsilon\right\} < 0. \tag{2.8}$$

Let us fix a number $\rho > 0$ satisfying the inequality $\rho\Lambda(\rho) \le \varepsilon/4\|\mu\|^+_\Lambda$. We divide the intervals $\{I_1, I_2, \dots I_N\}$ into two groups. The first group $\{I_n^{(1)}\}_n$ consists of intervals I_n such that

$$\min\{\mu(I_n) + M|I_n|\Lambda(|I_n|), 4\|\mu\|^+_\Lambda |I_n|\Lambda(|I_n|), \varepsilon\} = \mu(I_n) + M|I_n|\Lambda(|I_n|), \tag{2.9}$$

and the second one is $\{I_n^{(2)}\}_n = \{I_n\}_n \setminus \{I_n^{(1)}\}_n$.

Using these definitions and the fact that Λ is non-increasing, we rewrite inequality (2.8) as

$$\sum_n \mu(I_n^{(1)}) + M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) < -4\|\mu\|^+_\Lambda \sum_{n\,:\,|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) - \varepsilon\,\mathrm{Card}\{n : |I_n^{(2)}| \ge \rho\}. \tag{2.10}$$

Next we establish three properties of these families of intervals. From now on we assume that $M > 4\|\mu\|^+_\Lambda$.

(1) We have $\{I_n^{(2)} : |I_n^{(2)}| \ge \rho\} \ne \emptyset$. Otherwise, by (2.10), we would have

$$\begin{aligned}
0 &= \mu(\mathbb{T}) = \sum_n \mu(I_n^{(1)}) + \sum_n \mu(I_n^{(2)}) \\
&\le -M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) - 4\|\mu\|^+_\Lambda \sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) + \|\mu\|^+_\Lambda \sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \\
&\le -M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) - 3\|\mu\|^+_\Lambda \sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) < 0.
\end{aligned}$$

(2) We have $\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le 2\Lambda(\rho)$. To prove this relation, we notice first that for every simple covering $\{J_n\}_n$ of $\mathbb{T}$, we have

$$0 = \mu(\mathbb{T}) = \sum_n \mu(J_n) = \sum_n \mu(J_n)^+ - \sum_n \mu(J_n)^-,$$

and hence,

$$\sum_n |\mu(J_n)| = \sum_n \mu(J_n)^+ + \sum_n \mu(J_n)^- = 2\sum_n \mu(J_n)^+ \le 2\|\mu\|^+_\Lambda \sum_n |J_n|\Lambda(|J_n|).$$

Applying this to our simple covering, we get

$$\sum_n |\mu(I_n^{(1)})| + \sum_n |\mu(I_n^{(2)})| \le 2\|\mu\|_\Lambda^+ \sum_n \Big[|I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \Big],$$

and hence,

$$-\sum_n \mu(I_n^{(1)}) \le 2\|\mu\|_\Lambda^+ \sum_n \Big[|I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \Big].$$

Now, using (2.10) we obtain that

$$M\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + 4\|\mu\|_\Lambda^+ \sum_{|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le 2\|\mu\|_\Lambda^+ \sum_n \Big[|I_n^{(1)}|\Lambda(|I_n^{(1)}|) + |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \Big],$$

and hence,

$$\Big(M - 2\|\mu\|_\Lambda^+\Big)\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) \le 2\|\mu\|_\Lambda^+ \Bigg[\sum_{|I_n^{(2)}|\ge\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) - \sum_{|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \Bigg]. \quad (2.11)$$

As a consequence, we have

$$\sum_{|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le \sum_{|I_n^{(2)}|\ge\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|),$$

and, finally,

$$\sum_n |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le 2 \sum_{|I_n^{(2)}|\ge\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \le 2\sum_n |I_n^{(2)}|\Lambda(\rho) \le 2\Lambda(\rho).$$

(3) We have

$$\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) \le \frac{2\|\mu\|_\Lambda^+}{M - 2\|\mu\|_\Lambda^+} \cdot \Lambda(\rho).$$

This property follows immediately from (2.11).

We set $F_M = \bigcup_n \overline{I_n^{(1)}}$. Inequality (2.10) and the properties (1)–(3) show that

(i) $Entr_\Lambda(F_M) = O(1), \quad M \to \infty$,

(ii) $|F_M|\Lambda(|F_M|) \le \frac{2\|\mu\|_\Lambda^+}{M-2\|\mu\|_\Lambda^+} \cdot \Lambda(\rho)$,

(iii) $\mu(F_M) \le -4\|\mu\|_\Lambda^+ \Big[\sum_n |I_n^{(1)}|\Lambda(|I_n^{(1)}|) + \sum_{n\,:\,|I_n^{(2)}|<\rho} |I_n^{(2)}|\Lambda(|I_n^{(2)}|) \Big] - \varepsilon$.

By Lemma 2.12 there exists a subsequence $M_n \to +\infty$ such that $F_n^* := F_{M_n}$ (composed of a finite number of closed arcs) converge to a Λ-Carleson set F. More precisely, $F \subset {F_n^*}^\delta$ and $F_n^* \subset F^\delta$ for every fixed $\delta > 0$ and for sufficiently large n. Furthermore, (iii) yields

$$\mu(F_n^*) \le -4\|\mu\|_\Lambda^+ \Bigg[\sum_k |R_{k,n}|\Lambda(|R_{k,n}|) + \sum_{k:|L_{k,n}|<\rho} |L_{k,n}|\Lambda(|L_{k,n}|) \Bigg] - \varepsilon, \quad (2.12)$$

where $F_n^* = \bigsqcup_k R_{k,n}$ and $\mathbb{T} \setminus F_n^* = \bigsqcup_k L_{k,n}$.

It remains to show that

$$\mu_s(F) < 0.$$

Otherwise, if $\mu_s(F) = 0$, then by Proposition 2.6 we have

$$\lim_{\delta \to 0} \mu(F^\delta) = 0.$$

Modifying a bit the set F_n^*, if necessary, we obtain $\lim_{\delta\to 0} \mu(F_n^* \cap F^\delta) = 0$. Now we can choose a sequence $\delta_n > 0$ rapidly converging to 0 and a sequence $\{k_n\}$ rapidly converging to ∞ such that the sets F_n defined by

$$F_n = F_{k_n}^* \setminus F^{\delta_{n+1}} \subset F^{\delta_n} \setminus F^{\delta_{n+1}},$$

and consisting of a finite number of intervals $\{I_{k,n}\}_k$ satisfy the inequalities

$$\mu(F_n) \le -4\|\mu\|_\Lambda^+ \left[\sum_k |I_{k,n}|\Lambda(|I_{k,n}|) + \sum_k |J_{n,k}|\Lambda(|J_{n,k}|)\right] - \varepsilon/2, \tag{2.13}$$

where $\bigsqcup_k J_{n,k} = (F^{\delta_n} \setminus F^{\delta_{n+1}}) \setminus F_n =: G_n$.

We denote by $\mathcal{I}_n$, $\mathcal{J}_n$, and $\mathcal{K}_n$ the systems of intervals that form F_n, G_n, and F^{δ_n}, respectively. Furthermore, we denote by $\mathcal{I}_0$ be the system of intervals complementary to F^{δ_1}, and we put $\mathcal{S}_n = (\overset{n}{\underset{k=1}{\cup}} \mathcal{I}_k) \cup (\overset{n}{\underset{k=1}{\cup}} \mathcal{J}_n) \cup \mathcal{K}_{n+1}$. Summing up the estimates on $\mu(F_n)$ in (2.13) we obtain

$$\begin{aligned}
\sum_{I\in\mathcal{I}_0} |\mu(I)| + \sum_{I\in\mathcal{S}_n} |\mu(I)| &\ge \sum_{i=1}^n |\mu(F_i)| \\
&\ge 4\|\mu\|_\Lambda^+ \sum_{i=1}^n \left[\sum_k |I_{i,k}|\Lambda(|I_{i,k}|) + \sum_k |J_{i,k}|\Lambda(|J_{i,k}|)\right] + n\varepsilon/2 \\
&= 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{S}_n} |I|\Lambda(|I|) - 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|) + n\varepsilon/2 \\
&= 4\|\mu\|_\Lambda^+ \left[\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |I|\Lambda(|I|) - \sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|)\right] \\
&- 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{I}_0} |I|\Lambda(|I|) + n\varepsilon/2. \qquad (2.14)
\end{aligned}$$

Notice that

$$\sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|) \le \sum_{|J_k|<2\delta_{n+1}} |J_k|\Lambda(|J_k|) + 2\delta_{n+1}\Lambda(\delta_{n+1}) \cdot \mathrm{Card}\{k : |J_k| \ge 2\delta_{n+1}\},$$

where $\{J_k\}_k$, $|J_1| \geq |J_2| \geq \dots$ are the complementary arcs to the Λ-Carleson set F. Since $\lim_{t\to 0} t\Lambda(t) = 0$, we obtain that

$$\lim_{n\to+\infty} \sum_{I\in\mathcal{K}_{n+1}} |I|\Lambda(|I|) = 0.$$

Thus for sufficiently large n, (2.14) gives us the following relation

$$\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |\mu(I)| \geq 4\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |I|\Lambda(|I|)$$

where $\mathcal{S}_n \cup \mathcal{I}_0$ is a simple covering of the unit circle. However, since $\mu \in B_\Lambda^+$, we have

$$\sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |\mu(I)| = 2 \sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} \max(\mu(I), 0) \leq 2\|\mu\|_\Lambda^+ \sum_{I\in\mathcal{S}_n\cup\mathcal{I}_0} |I|\Lambda(|I|).$$

This contradiction completes the proof of the theorem.

2.2 Harmonic functions of restricted growth

Every bounded harmonic function can be represented via the Poisson integral of its boundary values. In the following theorem we show that a large class of real-valued harmonic functions in the unit disk $\mathbb{D}$ can be represented as the Poisson integrals of Λ-bounded premeasures. Before formulating the main result of this section, let us introduce some notations.

Definition 2.13 *Let f be a function in $C^1(\mathbb{T})$ and let $\mu \in B_\Lambda^+$. We define the integral of the function f with respect to μ by the formula*

$$\int_{\mathbb{T}} f\, d\mu = \int_0^{2\pi} f(e^{it})\, d\overset{\curlywedge}{\mu}(t).$$

In particular, we have

$$\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\mu(\theta) = -\int_0^{2\pi} \Big(\frac{\partial}{\partial\theta}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\Big)\overset{\curlywedge}{\mu}(\theta)\, d\theta.$$

Given a Λ-bounded premeasure μ we denote by $P[\mu]$ its Poisson integral :

$$P[\mu](z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\mu(\theta).$$

Proposition 2.14 *Let $\mu \in B_\Lambda^+$. The Poisson integral $P[\mu]$ satisfies the estimate*

$$P[\mu](z) \leq 10\|\mu\|_\Lambda^+ \Lambda(1-|z|), \qquad z \in \mathbb{D}.$$

Proof. It suffices to verify the estimate on the interval $(0,1)$. Let $0 < r < 1$. Then

$$\begin{aligned}
P[\mu](r) &= \int_0^{2\pi} \frac{1-r^2}{|e^{i\theta}-r|^2}\, d\mu(\theta) \\
&= -\int_0^{2\pi} \Big[\frac{\partial}{\partial\theta}\big(\frac{1-r^2}{|e^{i\theta}-r|^2}\big)\Big]\overset{\curlywedge}{\mu}(\theta)\, d\theta \\
&= \int_0^{2\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu(I_\theta)\, d\theta \\
&= \int_0^{\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu(I_\theta)\, d\theta - \int_\pi^0 -\frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu(I_{2\pi-\theta})\, d\theta \\
&= \int_0^{\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\Big[\mu(I_\theta)+\mu([-\theta,0))\Big]\, d\theta \\
&= \int_0^{\pi} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}\mu([-\theta,\theta))\, d\theta.
\end{aligned}$$

Integrating by parts and using the fact that Λ is decreasing and $t\Lambda(t)$ is increasing we get

$$P[\mu](r) \le \|\mu\|_\Lambda^+\Lambda(1-r)\Big[(1-r)\int_0^{\frac{1-r}{2}} \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}d\theta - \int_{\frac{1-r}{2}}^{\pi} 2\theta\Big[\frac{\partial}{\partial\theta}\big(\frac{1-r^2}{|e^{i\theta}-r|^2}\big)\Big]\, d\theta\Big]$$

$$\le \|\mu\|_\Lambda^+\Lambda(1-r)\Big[2(1-r)^3\int_0^{\frac{1-r}{2}} \frac{d\theta}{(1-r)^4} + \frac{(1-r)(1-r^2)}{(1-r)^2} + 2\int_0^{\pi} \frac{1-r^2}{|e^{i\theta}-r|^2}d\theta\Big]$$

$$\le 10\|\mu\|_\Lambda^+\Lambda(1-r).$$

□

The following theorem is stated by Korenblum in [36, Theorem 1, p. 543] without proof, in a more general situation.

Theorem 2.15 *Let h be a real-valued harmonic function on the unit disk such that $h(0) = 0$ and*

$$h(z) = O(\Lambda(1-|z|)), \qquad |z| \to 1,\ z \in \mathbb{D}.$$

Then the following statements hold.

1. *For every open arc I of the unit circle $\mathbb{T}$ the following limit exists :*

$$\mu(I) = \lim_{r\to 1^-}\mu_r(I) = \lim_{r\to 1^-}\int_I h(r\xi)\,|d\xi| < \infty.$$

2. μ is a Λ-bounded premeasure.

3. The function h is the Poisson integral of the premeasure μ :

$$h(z)=\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu(\theta),\qquad z\in\mathbb{D}.$$

Proof. Let

$$h(re^{i\theta})=\sum_{n=-\infty}^{+\infty}a_n r^{|n|}e^{in\theta}.$$

Since $a_0=h(0)=0$, we have

$$\int_0^{2\pi}h^+(re^{i\theta})\,d\theta=\int_0^{2\pi}h^-(re^{i\theta})\,d\theta=\frac{1}{2}\int_0^{2\pi}|h(re^{i\theta})|\,d\theta.$$

Furthermore,

$$\begin{aligned}|a_n| &= \Big|\frac{r^{-|n|}}{2\pi}\int_0^{2\pi}h(re^{i\theta})e^{-in\theta}\,d\theta\Big|\\ &\le \frac{r^{-|n|}}{2\pi}\int_0^{2\pi}|h(re^{i\theta})|\,d\theta=\frac{r^{-|n|}}{\pi}\int_0^{2\pi}h^+(re^{i\theta})\,d\theta\\ &\le Cr^{-|n|}\Lambda(1-r)\\ &\le C_1\Lambda\Big(\frac{1}{|n|}\Big),\qquad \frac{1}{|n|}=1-r,\ n\in\mathbb{Z}\setminus\{-1,0,1\}.\end{aligned}\tag{2.15}$$

Let $I=\{e^{i\theta}:\alpha\le\theta\le\beta\}$ be an arc of $\mathbb{T}$, $\tau=\beta-\alpha$. For $\theta\in[\alpha,\beta]$ we define

$$t(\theta)=\min\{\theta-\alpha,\beta-\theta\},\qquad \eta(\theta)=\frac{1}{\tau}(\beta-\theta)(\theta-\alpha).$$

Then

$$\frac{1}{2}t(\theta)\le\eta(\theta)\le t(\theta),\qquad |\eta'(\theta)|\le 1,\qquad \eta''(\theta)=\frac{-2}{\tau},\qquad \theta\in[\alpha,\beta].$$

Given $p>2$ we introduce the function $q(\theta)=1-\eta(\theta)^p$ satisfying the following properties :

$$|q'(\theta)|\le p\eta(\theta)^{p-1},\qquad |q''(\theta)|\le p^2\eta(\theta)^{p-2},\qquad \theta\in(\alpha,\beta).$$

Integrating by parts we obtain for $|n|\ge 1$ and $\tau<1$ that

$$\begin{aligned}\Big|\int_\alpha^\beta(1-q(\theta)^{|n|})e^{in\theta}\,d\theta\Big| &= \frac{1}{|n|}\Big|\int_\alpha^\beta |n|q(\theta)^{|n|-1}q'(\theta)e^{in\theta}\,d\theta\Big|\\ &\le \frac{|n|-1}{|n|}\int_\alpha^\beta q(\theta)^{|n|-2}|q'(\theta)|^2\,d\theta+\frac{1}{|n|}\int_\alpha^\beta q(\theta)^{|n|-1}|q''(\theta)|\,d\theta\\ &\le 2p^2\int_0^{\tau/2}\Big(1-\Big[\frac{t}{2}\Big]^p\Big)^{|n|-2}t^{2p-2}\,dt+\frac{2p^2}{|n|}\int_0^{\tau/2}\Big(1-\Big[\frac{t}{2}\Big]^p\Big)^{|n|-1}t^{p-2}\,dt\\ &\le C_p\Big[\int_0^{\tau/4}\big(1-t^p\big)^{|n|-2}t^{2p-2}\,dt+\frac{1}{|n|}\int_0^{\tau/4}\big(1-t^p\big)^{|n|-1}t^{p-2}\,dt\Big],\end{aligned}$$

and, hence,

$$\begin{aligned}\Big|\int_\alpha^\beta (1-q(\theta)^{|n|})e^{in\theta}\,d\theta\Big| &\le C_{1,p}\tau \max_{0\le t\le 1}\Big\{\big(1-t^p\big)^{|n|-2}t^{2p-2}+\frac{1}{|n|}\big(1-t^p\big)^{|n|-1}t^{p-2}\Big\}\\ &\le C_{2,p}\tau|n|^{-2(1-\frac{1}{p})}.\end{aligned}$$

On the other hand, we have

$$\frac{1}{2\pi}\int_I h(r\xi)\,|d\xi| = \frac{1}{2\pi}\int_\alpha^\beta h(rq(\theta)e^{i\theta})\,d\theta + \frac{1}{2\pi}\int_\alpha^\beta \big[h(re^{i\theta})-h(rq(\theta)e^{i\theta})\big]\,d\theta.$$

By (2.15), we obtain

$$\begin{aligned}\Big|\frac{1}{2\pi}\int_\alpha^\beta \big[h(re^{i\theta})-h(rq(\theta)e^{i\theta})\big]\,d\theta\Big| &\le \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}|a_n|\Big|\int_\alpha^\beta r^{|n|}(1-q(\theta)^{|n|})e^{in\theta}\,d\theta\Big|\\ &\le C_{3,p}\tau\sum_{n\in\mathbb{Z}}|a_n|(|n|+1)^{-2(1-\frac{1}{p})}\\ &\le C_{4,p}\tau\sum_{n\in\mathbb{Z}}\Lambda\Big(\frac{1}{\max(|n|,1)}\Big)(|n|+1)^{-2(1-\frac{1}{p})}.\end{aligned}$$

Therefore, if $t\mapsto t^\alpha\Lambda(t)$ increase, and

$$\alpha+\frac{2}{p}<1, \tag{2.16}$$

then

$$\Big|\frac{1}{2\pi}\int_\alpha^\beta \big[h(re^{i\theta})-h(rq(\theta)e^{i\theta})\big]\,d\theta\Big| \le C_{5,p}\tau.$$

Since $\Lambda(x^p)\le C_p\Lambda(x)$, we obtain

$$\begin{aligned}\Big|\frac{1}{2\pi}\int_\alpha^\beta h(rq(\theta)e^{i\theta})\,d\theta\Big| &\le C\int_\alpha^\beta \Lambda(1-q(\theta))\,d\theta\\ &\le C\int_\alpha^\beta \Lambda\Big(\frac{t(\theta)}{2}\Big)\,d\theta\\ &\le C_1\int_0^{\tau/4}\Lambda(t)\,dt\\ &= C_1\int_0^{\tau/4} t^{-\alpha}t^\alpha\Lambda(t)\,dt\\ &\le C_2\tau^\alpha\Lambda(\tau)\int_0^{\tau/4}t^{-\alpha}\,dt\\ &= C_3\tau\Lambda(\tau).\end{aligned}$$

Hence,

$$\mu_r(I) \le C|I|\Lambda(|I|)$$

for some C independent of I.

Given $r \in (0,1)$, we define $h_r(z) = h(rz)$. The h_r is the Poisson integral of $d\mu_r = h_r(e^{i\theta})\,d\theta$:

$$h_r(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu_r(\theta)$$

The set $\{\mu_r : r \in (0,1)\}$ is a uniformly Λ-bounded family of premeasures. Using a Helly-type theorem [31, Theorem 1, p. 204], we can find a sequence of premeasures $\mu_{r_n} \in B^+_\Lambda$ converging weakly to a Λ-bounded premeasure μ as $n \to \infty$, $\lim_{n\to\infty} r_n = 1$. Then

$$\mu(I) \le C|I|\Lambda(|I|)$$

for every arc I, and

$$h_{r_n}(z) = -\int_0^{2\pi} \frac{\partial}{\partial\theta}\Big(\frac{1-|z|^2}{|e^{i\theta}-z|^2}\Big)\overset{\lambda}{\mu}_n(\theta)\,d\theta.$$

Passing to the limit we conclude that

$$h(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\mu(\theta).$$

□

2.3 Cyclic vectors

Given a Λ-bounded premeasure μ, we consider the corresponding analytic fuction

$$f_\mu(z) = \exp\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta). \tag{2.17}$$

If $\tilde\mu$ is a positive singular measure on the circle $\mathbb{T}$, we denote by $S_{\tilde\mu}$ the associated singular inner function. Notice that in this case $\mu = \tilde\mu(\mathbb{T})m - \tilde\mu$ is a premeasure, and we have $S_{\tilde\mu} = f_\mu/S_{\tilde\mu}(0)$; m is (normalized) Lebesgue measure.

Let f be a zero-free function in $\mathcal{A}^{-\infty}_\Lambda$ such that $f(0)=1$. According to Theorem 2.15, there is a premeasure $\mu_f \in B^+_\Lambda$ such that

$$f(z) = \exp\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu_f(\theta).$$

The following result follows immediately from Theorem 2.8.

Theorem 2.16 *Let $f \in \mathcal{A}^{-\infty}_\Lambda$ be a zero-free function such that $f(0)=1$. If $(\mu_f)_s \equiv 0$, then f is cyclic in $\mathcal{A}^{-\infty}_\Lambda$.*

Proof. Suppose that $(\mu_f)_s \equiv 0$. By theorem 2.8, μ_f is Λ-absolutely continuous. Let $\{\mu_n\}_{n\geq 1}$ be a sequence of Λ-bounded premeasures from Definition 2.7. We set

$$g_n(z) = \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu_n(\theta), \qquad z \in \mathbb{D}.$$

By Proposition 2.14, $g_n \in \mathcal{A}_\Lambda^{-\infty}$, and

$$\begin{aligned} f(z)g_n(z) &= \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d(\mu_f+\mu_n)(\theta) \\ &= \exp\Big[-\int_0^{2\pi} \frac{\partial}{\partial\theta}\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big)[\overset{\curlywedge}{\mu}_n(\theta) - \overset{\curlywedge}{\mu}(\theta)]\, d\theta\Big] \\ &= \exp\Big[-\int_0^{2\pi} \frac{\partial}{\partial\theta}\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big)\big[\mu(I_\theta) + \mu_n(I_\theta)\big]\, d\theta\Big]. \end{aligned}$$

Again by Definition 2.7, we obtain that $f(z)g_n(z) \to 1$ uniformly on compact subsets of unit disk $\mathbb{D}$. This yields that $fg_n \to 1$ in $\mathcal{A}_\Lambda^{-\infty}$ as $n \to \infty$. □

From now on, we deal with the statements converse to Theorem 2.16. We'll establish two results valid for different growth ranges of the majorant Λ. More precisely, we consider the following growth and regularity assumptions :

$$\text{for every } c>0, \text{ the function } x \mapsto \exp\big[c\Lambda(1/x)\big] \text{ is concave for large } x, \tag{C1}$$

$$\lim_{t\to 0} \frac{\Lambda(t)}{\log(1/t)} = \infty. \tag{C2}$$

Examples of majorants Λ satisfying condition (C1) include

$$(\log(1/x))^p, \quad 0<p<1, \quad \text{and} \quad \log(\log(1/x)), \quad x \to 0.$$

Examples of majorants Λ satisfying condition (C2) include

$$(\log(1/x))^p, \quad p>1.$$

Thus, we consider majorants which grow less rapidly than the Korenblum majorant ($\Lambda(x) = \log(1/x)$) in Case 1 or more rapidly than the Korenblum majorant in Case 2.

2.3.1 Weights Λ satisfying condition (C1)

We start with the following observation :

$$\Lambda(t) = o(\log 1/t), \qquad t \to 0.$$

Next we pass to some notations and auxiliary lemmas. Given a function f in $L^1(\mathbb{T})$, we denote by $P[f]$ its Poisson transform,

$$P[f](z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|} f(e^{i\theta})\, d\theta, \qquad z \in \mathbb{D}.$$

Denote by $A(\mathbb{D})$ the disk-algebra, i.e., the algebra of functions continuous on the closed unit disk and holomorphic in $\mathbb{D}$. A positive continuous increasing function ω on $[0,\infty)$ is said to be a modulus of continuity if $\omega(0)=0$, $t\mapsto \omega(t)/t$ decreases near 0, and $\lim_{t\to 0}\omega(t)/t=\infty$. Given a modulus of continuity ω, we consider the Lipschitz space $\mathrm{Lip}_\omega(\mathbb{T})$ defined by

$$\mathrm{Lip}_\omega(\mathbb{T})=\{f\in C(\mathbb{T}):|f(\xi)-f(\zeta)|\le C(f)\omega(|\xi-\zeta|)\}.$$

Since the function $t\mapsto \exp[2\Lambda(1/t)]$ is concave for large t, and $\Lambda(t)=o(\log(1/t))$, $t\to 0$, we can apply a result of Kellay [26, Lemma 3.1], to get a non-negative summable function Ω_Λ on $[0,1]$ such that

$$e^{2\Lambda(\frac{1}{n+1})}-e^{2\Lambda(\frac{1}{n})}\asymp\int_{1-\frac{1}{n}}^{1}\Omega_\Lambda(t)dt,\qquad n\ge 1.$$

Next we consider the Hilbert space $L^2_{\Omega_\Lambda}(\mathbb{T})$ of the functions $f\in L^2(\mathbb{T})$ such that

$$\|f\|^2_{\Omega_\Lambda}=|P[f](0)|^2+\int_{\mathbb{D}}\frac{P[|f|^2](z)-|P[f](z)|^2}{1-|z|^2}\Omega_\Lambda(|z|)\,dA(z)<\infty,$$

where dA denote the normalized area measure. We need the following lemma.

Lemma 2.17 *Under our conditions on Λ and Ω_Λ, we have*

1. $\|f\|^2_{\Omega_\Lambda}\asymp\sum_{n\in\mathbb{Z}}|\hat f(n)|^2e^{2\Lambda(1/n)}$, $\quad f\in L^2_{\Omega_\Lambda}(\mathbb{T})$,
2. *the functions* $\exp(-c\Lambda(t))$ *are moduli of continuity for* $c>0$,
3. *for some positive a, the function* $\rho(t)=\exp(-\frac{3}{2a}\Lambda(t))$ *satisfies the property*

$$\mathrm{Lip}_\rho(\mathbb{T})\subset L^2_{\Omega_\Lambda}(\mathbb{T}).$$

For the first statement see [11, Lemma 6.1] (where it is attributed to Aleman [1]) ; the second statement is [11, Lemma 8.4] ; the third statement follows from [11, Lemmas 6.2 and 6.3].

Recall that

$$\mathcal{A}^{-1}_\Lambda=\{f\in \mathrm{Hol}(\mathbb{D}):|f(z)|\le C(f)\exp(\Lambda(1-|z|))\}.$$

Lemma 2.18 *Under our conditions on Λ, there exists a positive number c such that*

$$P_+\mathrm{Lip}_{e^{-c\Lambda}}(\mathbb{T})\subset(\mathcal{A}^{-1}_\Lambda)^*$$

via the Cauchy duality

$$\langle f,g\rangle=\sum_{n\ge 0}a_n\overline{\widehat{g}(n)},$$

where $f(z)=\sum_{n\ge0}a_nz^n\in\mathcal{A}^{-1}_\Lambda$, $g\in\mathrm{Lip}_{e^{-c\Lambda}}(\mathbb{T})$, *and P_+ is the orthogonal projector from* $\mathrm{L}^2(\mathbb{T})$ *onto* $\mathrm{H}^2(\mathbb{D})$.

Proof. Denote

$$L^2_\Lambda(\mathbb{D}) = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 |\Lambda'(1-|z|)| e^{-2\Lambda(1-|z|)}\, dA(z) < +\infty \Big\},$$

and

$$\mathcal{B}^2_\Lambda = \big\{ f(z) = \sum_{n\ge 0} a_n z^n : |a_0|^2 + \sum_{n>0} |a_n|^2 e^{-2\Lambda(1/n)} < \infty \big\}.$$

Let us prove that

$$L^2_\Lambda(\mathbb{D}) = \mathcal{B}^2_\Lambda. \tag{2.18}$$

To verify this equality, it suffices sufficient to check that

$$e^{-2\Lambda(1/n)} \asymp \int_0^1 r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)}\, dr.$$

In fact,

$$\int_{1-1/n}^1 r^{2n+1} |\Lambda^{'}(1-r)| e^{-2\Lambda(1-r)}\, dr \asymp \int_{1-1/n}^1 |\Lambda^{'}(1-r)| e^{-2\Lambda(1-r)}\, dr \asymp e^{-2\Lambda(\frac{1}{n})}, \qquad n \ge 1.$$

On the other hand,

$$\begin{aligned}
\int_0^{1-1/n} r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)}\, dr &= -\int_0^{1-1/n} r^{2n+1}\, de^{-2\Lambda(1-r)} \\
&\asymp -e^{-2\Lambda(1/n)} + (2n+1) \int_0^{1-1/n} r^{2n} e^{-2\Lambda(1-r)} dr \\
&\asymp n \sum_{k=1}^n e^{-2n/k} e^{-2\Lambda(1/k)} \frac{1}{k^2}.
\end{aligned}$$

Since the function $\exp\big[2\Lambda(1/x)\big]$ is concave, we have $e^{2\Lambda(1/k)} \ge \frac{k}{n} e^{2\Lambda(1/n)}$, and hence,

$$e^{-2\Lambda(1/k)} \le \frac{n}{k} e^{-2\Lambda(1/n)}.$$

Therefore,

$$\int_0^{1-1/n} r^{2n+1} |\Lambda'(1-r)| e^{-2\Lambda(1-r)}\, dr \le C n^2 e^{-2\Lambda(1/n)} \sum_{k=1}^n e^{-2n/k} \frac{1}{k^3} \asymp e^{-2\Lambda(1/n)},$$

and (2.18) follows.

Since $\mathcal{A}^{-1}_\Lambda \subset L^2_\Lambda(\mathbb{D})$, we have $(\mathcal{B}^2_\Lambda)^* \subset (\mathcal{A}^{-1}_\Lambda)^*$. By Lemma 2.17, we have $P_+\mathrm{Lip}_\rho(\mathbb{T}) \subset (\mathcal{B}^2_\Lambda)^*$. Thus,

$$P_+\mathrm{Lip}_\rho(\mathbb{T}) \subset (\mathcal{A}^{-1}_\Lambda)^*.$$

□

Lemma 2.19 *Let $f \in \mathcal{A}_\Lambda^{-n}$ for some $n > 0$. The function f is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if there exists $m > n$ such that f is cyclic in $\mathcal{A}_\Lambda^{-m}$.*

Proof. Notice that the space $\mathcal{A}_\Lambda^{-\infty}$ is endowed with the inductive limit topology induced by the spaces $\mathcal{A}_\Lambda^{-N}$. A sequence $\{f_n\}_n \in \mathcal{A}_\Lambda^{-\infty}$ converges to $g \in \mathcal{A}_\Lambda^{-\infty}$ if and only if there exists $N > 0$ such that all f_n and g belong to $\mathcal{A}_\Lambda^{-N}$, and $\lim_{n\to+\infty} \|f_n - g\|_{\mathcal{A}_\Lambda^{-N}} = 0$. The statement of the lemma follows. □

Theorem 2.20 *Let $\mu \in B_\Lambda^+$, and let the majorant Λ satisfy condition* (C1). *Then the function f_μ is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if $\mu_s \equiv 0$.*

Proof. Suppose that the Λ-singular part μ_s of μ is non-trivial. There exists a Λ-Carleson set $F \subset \mathbb{T}$ such that $-\infty < \mu_s(F) < 0$. We set $\nu = -\mu_s \big| F$. By a theorem of Shirokov [46, Theorem 9, pp.137,139], there exists an outer function φ such that

$$\varphi \in \mathrm{Lip}_\rho(\mathbb{T}) \cap \mathrm{H}^\infty(\mathbb{D}), \qquad \varphi S_\nu \in \mathrm{Lip}_\rho(\mathbb{T}) \cap \mathrm{H}^\infty(\mathbb{D}),$$

and the zero set of the function φ coincides with F. Next, for $\xi, \theta \in [0, 2\pi]$ we have

$$\begin{aligned} |\varphi\overline{S_\nu}(e^{i\xi}) - \varphi\overline{S_\nu}(e^{i\theta})| &= |\varphi(e^{i\xi})S_\nu(e^{i\theta}) - \varphi(e^{i\theta})S_\nu(e^{i\xi})| \\ &\leq |\big(\varphi(e^{i\xi}) - \varphi(e^{i\theta})\big)S_\nu(e^{i\theta})| + |\big(\varphi(e^{i\theta}) - \varphi(e^{i\xi})\big)S_\nu(e^{i\xi})| \\ &+ |(\varphi S_\nu)(e^{i\theta}) - (\varphi S_\nu)(e^{i\xi})|, \end{aligned}$$

and hence,

$$\varphi\overline{S_\nu} \in \mathrm{Lip}_\rho(\mathbb{T}).$$

Set $g = P_+\big(\overline{z\varphi}S_\nu\big)$. Since $\varphi\overline{S_\nu} \in \mathrm{Lip}_\rho(\mathbb{T})$, we have $g \in (\mathcal{A}_\Lambda^{-1})^*$. Consider the following linear functional on $\mathcal{A}_\Lambda^{-1}$:

$$L_g(f) = \langle f, g\rangle = \sum_{n\geq 0} a_n \overline{\widehat{g}(n)}, \qquad f(z) = \sum_{n\geq 0} a_n z^n \in \mathcal{A}_\Lambda^{-1}.$$

Suppose that $L_g = 0$. Then, for every $n \geq 0$ we have

$$\begin{aligned} 0 &= L_g(z^n) \\ &= \int_0^{2\pi} e^{in\theta}\overline{g(e^{i\theta})}\frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{i(n+1)\theta}\frac{\varphi(e^{i\theta})}{S_\nu(e^{i\theta})}\frac{d\theta}{2\pi}. \end{aligned}$$

We conclude that $\varphi/S_\nu \in \mathrm{H}^\infty(\mathbb{D})$, which is impossible. Thus, $L_g \neq 0$.

On the other hand we have, for every $n \geq 0$,

$$\begin{aligned} L_g(z^n S_\nu) &= \int_0^{2\pi} e^{in\theta} S_\nu(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{in\theta} S_\nu(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} e^{i(n+1)\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi} \\ &= 0. \end{aligned}$$

Thus, $g \perp [f_\mu]_{\mathcal{A}_\Lambda^{-1}}$ which implies that the function f_μ is not cyclic in $\mathcal{A}_\Lambda^{-1}$. By Lemma 2.19, f_μ is not cyclic in $\mathcal{A}_\Lambda^{-\infty}$. $\square$

2.3.2 Weights Λ satisfying condition (C2)

We start with an elementary consequence of the Cauchy formula.

Lemma 2.21 *Let $f(z) = \sum_{n\geq 0} a_n z^n$ be an analytic function in $\mathbb{D}$. If $f \in \mathcal{A}_\Lambda^{-\infty}$, then there exists $C > 0$ such that*

$$|a_n| = O(\exp[C\Lambda(\frac{1}{n})]) \quad \textit{as} \quad n \to +\infty.$$

Theorem 2.22 *Let $\mu \in B_\Lambda^+$, and let the majorant Λ satisfy condition* (C2)*. Then the function f_μ is cyclic in $\mathcal{A}_\Lambda^{-\infty}$ if and only if $\mu_s \equiv 0$.*

Proof. We define

$$\mathcal{A}_\Lambda^\infty = \bigcap_{c<\infty} \left\{ g \in \mathrm{Hol}(\mathbb{D}) \cap C^\infty(\bar{\mathbb{D}}) : |\widehat{f}(n)| = O(\exp[-c\Lambda(\frac{1}{n})]) \right\},$$

and, using Lemma 2.21, we obtain that $\mathcal{A}_\Lambda^\infty \subset (\mathcal{A}_\Lambda^{-\infty})^*$ via the Cauchy duality

$$\langle f, g \rangle = \sum_{n\geq 0} \widehat{f}(n) \overline{\widehat{g}(n)} = \lim_{r\to 1} \int_0^{2\pi} f(r\xi) \overline{g(\xi)} d\xi, \qquad f \in \mathcal{A}_\Lambda^{-\infty}, \quad g \in \mathcal{A}_\Lambda^\infty.$$

Suppose that the Λ-singular part μ_s of μ is nonzero. Then there exists a Λ-Carleson set $F \subset \mathbb{T}$ such that $-\infty < \mu_s(F) < 0$. We set $\sigma = \mu_s | F$. By a theorem of Bourhim, El-Fallah, and Kellay [11, Theorem 5.3] (extending a result of Taylor and Williams), there exist an outer function $\varphi \in \mathcal{A}_\Lambda^\infty$ such that the zero set of φ and of all its derivatives coincides exactly with the set F, a function $\widetilde{\Lambda}$ such that

$$\Lambda(t) = o(\widetilde{\Lambda}(t)), \qquad t \to 0, \tag{2.19}$$

and a positive constant B such that

$$|\varphi^{(n)}(z)| \le n! B^n e^{\widetilde{\Lambda}^*(n)}, \qquad n \ge 0,\ z \in \mathbb{D}, \tag{2.20}$$

where $\widetilde{\Lambda}^*(n) = \sup_{x>0}\bigl\{nx - \widetilde{\Lambda}(e^{-x/2})\bigr\}$.

We set

$$\Psi = \varphi \overline{S_\sigma}.$$

For some positive D we have

$$|S_\sigma^{(n)}(z)| \le \frac{D^n n!}{\operatorname{dist}(z,F)^{2n}}, \qquad z \in \mathbb{D},\ n \ge 0. \tag{2.21}$$

By the Taylor formula, for every $n, k \ge 0$, we have

$$|\varphi^{(n)}(z)| \le \frac{1}{k!} \operatorname{dist}(z,F)^k \max_{w\in\mathbb{D}} |\varphi^{(n+k)}(w)|, \qquad z \in \mathbb{D}. \tag{2.22}$$

Next, integrating by parts, for every $n \ne 0$, $k \ge 0$ we obtain

$$|\widehat{\Psi}(n)| = \bigl|\widehat{(\varphi\overline{S_\sigma})}(n)\bigr| = \frac{1}{2\pi}\Bigl|\int_0^{2\pi} \frac{(\varphi\overline{S_\sigma})^{(k)}(e^{it})}{n^k} e^{-int}\, dt\Bigr|.$$

Applying the Leibniz formula and estimates (2.20)–(2.22), we obtain for $n \ge 1$ that

$$\begin{aligned}
\bigl|\widehat{\Psi}(n)\bigr| &\le \inf_{k\ge 0}\Bigl\{\frac{1}{n^k}\max_{t\in[0,2\pi]}\bigl|(\varphi\overline{S_\sigma})^{(k)}(e^{it})\bigr|\Bigr\} \\
&\le \inf_{k\ge 0}\Bigl\{\frac{1}{n^k}\sum_{s=0}^{k} C_k^s \max_{t\in[0,2\pi]}|S_\sigma^{(s)}(e^{it})| \max_{t\in[0,2\pi]}|\varphi^{(k-s)}(e^{it})|\Bigr\} \\
&\le \inf_{k\ge 0}\Bigl\{\frac{1}{n^k}\sum_{s=0}^{k} C_k^s D^s s! \frac{1}{(2s)!}(k+s)! B^{k+s} e^{\widetilde{\Lambda}^*(k+s)}\Bigr\} \\
&\le \inf_{k\ge 0}\Bigl\{e^{\widetilde{\Lambda}^*(2k)}\Bigl(\frac{B^2 D}{n}\Bigr)^k \sum_{s=0}^{k}\frac{(k+s)!k!}{(2s)!(k-s)!}\Bigr\} \\
&\le \inf_{k\ge 0}\Bigl\{k! e^{\widetilde{\Lambda}^*(2k)}\Bigl(\frac{4B^2 D}{n}\Bigr)^k\Bigr\} \\
&\le \inf_{k\ge 0} k!\Bigl\{\Bigl(\frac{4B^2 D}{n}\Bigr)^k \sup_{0<t<1}\Bigl\{e^{-\widetilde{\Lambda}(t^{1/4})} t^{-k}\Bigr\}\Bigr\}.
\end{aligned}$$

By property (2.19), for every $C > 0$ there exists a positive number K such that

$$e^{-\widetilde{\Lambda}(t^{1/4})} \le K e^{-\Lambda(Ct)}, \qquad t \in (0,1).$$

We take $C = \frac{1}{8B^2D}$, and obtain for $n \neq 0$ that

$$\begin{aligned} \big|\widehat{\Psi}(n)\big| &\leq K \inf_{k\geq 0}\Big\{\Big(\frac{4B^2D}{n}\Big)^k k! \sup_{0<t<1} \frac{e^{-\Lambda(Ct)}}{t^k}\Big\} \\ &\leq K_1 \inf_{k\geq 0}\Big\{(2n)^{-k} k! \sup_{0<t<1} \frac{e^{-\Lambda(t)}}{t^k}\Big\}. \end{aligned}$$

Finally, using [28, Lemma 6.5] (see also [11, Lemma 8.3]), we get

$$|\widehat{\Psi}(n)| = O\big(e^{-\Lambda(1/n)}\big), \qquad |n| \to \infty.$$

Thus, the function $g = P_+\big(\overline{z\varphi}S_\sigma\big)$ belongs to $(\mathcal{A}_\Lambda^{-1})^*$. Now we obtain that f_μ is not cyclic using the same argument as that at the end of Case 1. This concludes the proof of the theorem. □

Theorems 2.20 and 2.22 together give a positive answer to a conjecture by Deninger [14, Conjecture 42].

We complete this section by two examples that show how the cyclicity property of a fixed function changes in a scale of $\mathcal{A}_\Lambda^{-\infty}$ spaces.

Example 2.23 *Let* $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, *and let* $0 < \alpha_0 < 1$. *There exists a singular inner function* S_μ *such that*

$$S_\mu \textit{ is cyclic in } \mathcal{A}_{\Lambda_\alpha}^{-\infty} \iff \alpha > \alpha_0.$$

Construction. We start by defining a Cantor type set and the corresponding canonical measure. Let $\{m_k\}_{k\geq 1}$ be a sequence of natural numbers. Set $M_k = \sum_{1\leq s\leq k} m_s$, and assume that

$$M_k \asymp m_k, \qquad k \to \infty. \tag{2.23}$$

Consider the following iterative procedure. Set $\mathcal{I}_0 = [0,1]$. On the step $n \geq 1$ the set $\mathcal{I}_{n-1}$ consist of several intervals I. We divide each I into 2^{m_n+1} equal subintervals and replace it by the union of every second interval in this division. The union of all such groups is $\mathcal{I}_n$. Correspondingly, $\mathcal{I}_n$ consists of 2^{M_n} intervals ; each of them is of length 2^{-n-M_n}. Next, we consider the probabilistic measure μ_n equidistributed on $\mathcal{I}_n$. Finally, we set $E = \cap_{n\geq 1}\mathcal{I}_n$, and define by μ the weak limit of the measures μ_n.

Now we estimate the Λ_α-entropy of E :

$$Entr_{\Lambda_\alpha}(\mathcal{I}_n) \asymp \sum_{1\leq k\leq n} 2^{M_k}\cdot 2^{-k-M_k}\cdot \Lambda_\alpha(2^{-k-M_k}) \asymp \sum_{1\leq k\leq n} 2^{-k}\cdot m_k^\alpha, \qquad n\to\infty.$$

Thus, if

$$\sum_{n\geq 1} 2^{-n}\cdot m_n^{\alpha_0} < \infty, \tag{2.24}$$

then $Entr_{\Lambda_{\alpha_0}}(E) < \infty$. By Theorem 2.20, S_μ is not cyclic in $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$ for $\alpha \leq \alpha_0$.

Next we estimate the modulus of continuity of the measure μ,

$$\omega_\mu(t) = \sup_{|I|=t} \mu(I).$$

Assume that

$$A_{j+1} = 2^{-(j+1)-M_{j+1}} \le |I| < A_j = 2^{-j-M_j},$$

and that I intersects with one of the intervals I_j that constitute $\mathcal{I}_j$. Then

$$\mu(I) \le 4\frac{|I|}{A_j}\mu(I_j) = 4|I|2^{j+M_j}2^{-M_j} = 4|I|2^j.$$

Thus, if

$$2^j \le C(\log(1/A_j))^\alpha \asymp m_j^\alpha, \qquad j \ge 1,\, \alpha_0 < \alpha < 1, \tag{2.25}$$

then

$$\omega_\mu(t) \le Ct(\log(1/t))^\alpha.$$

By [3, Corollary B], we have $\mu(F) = 0$ for any Λ_α-Carleson set F, $\alpha_0 < \alpha < 1$. Again by Theorem 2.20, S_μ is cyclic in $\mathcal{A}^{-\infty}_{\Lambda_\alpha}$ for $\alpha > \alpha_0$. It remains to fix $\{m_k\}_{k\ge1}$ satisfying (2.23)–(2.25). The choice $m_k = 2^{k/\alpha_0}k^{-2/\alpha_0}$ works. $\square$

Of course, instead of Theorem 2.20 we could use here [11, Theorem 7.1].

Example 2.24 *Let* $\Lambda_\alpha(x) = (\log(1/x))^\alpha$, $0 < \alpha < 1$, *and let* $0 < \alpha_0 < 1$. *There exists a premeasure* μ *such that* μ_s *is infinite,*

$$f_\mu \text{ is cyclic in } \mathcal{A}^{-\infty}_{\Lambda_\alpha} \iff \alpha > \alpha_0,$$

where f_μ *is defined by* (2.17).

It looks like the subspaces $[f_\mu]_{\mathcal{A}^{-\infty}_{\Lambda_\alpha}}$, $\alpha \le \alpha_0$, contain no nonzero Nevanlinna class functions. For a detailed discussion on Nevanlinna class generated invariant subspaces in the Bergman space (and in the Korenblum space) see [23].

For $\alpha \le \alpha_0$, instead of Theorem 2.20 we could once again use here [11, Theorem 7.1].

Construction. We use the measure μ constructed in Example 2.23.

Choose a decreasing sequence u_k of positive numbers such that

$$\sum_{k\ge1} u_k = 1, \qquad \sum_{k\ge1} v_k = +\infty,$$

where $v_k = u_k \log\log(1/u_k) > 0$, $k \ge 1$.

Given a Borel set $B \subset B^0 = [0,1]$, denote

$$B_k = \{u_k t + \sum_{j=1}^{k-1} u_j : t \in B\} \subset [0,1],$$

and define measures ν_k supported by B_k^0 by

$$\nu_k(B_k) = \frac{v_k}{u_k} m(B_k) - v_k \mu(B),$$

where $m(B_k)$ is Lebesgue measure of B_k.

We set

$$\nu = \sum_{k \geq 1} \nu_k.$$

Then $\nu(B_k^0) = \nu_k(B_k^0) = 0$, $k \geq 1$, and ν is a premeasure.

Since

$$v_k \leq C(\alpha) u_k \Lambda_\alpha(u_k), \qquad 0 < \alpha < 1,$$

ν is a Λ_α-bounded premeasure for $\alpha \in (0,1)$.

Furthermore, as above, by Theorem 2.20, f_ν is not cyclic in $\mathcal{A}^{-\infty}_{\Lambda_\alpha}$ for $\alpha \leq \alpha_0$.

Next, we estimate

$$\omega_\nu(t) = \sup_{|I|=t} |\nu(I)|.$$

As in Example 2.23, if $j, k \geq 1$ and

$$u_k A_{j+1} \leq |I| < u_k A_j,$$

then

$$\frac{|\nu(I)|}{|I|} \leq C \cdot 2^j \cdot \frac{v_k}{u_k}. \tag{2.26}$$

Now we verify that

$$\omega_\nu(t) \leq Ct(\log(1/t))^\alpha, \qquad \alpha_0 < \alpha < 1. \tag{2.27}$$

Fix $\alpha \in (\alpha_0, 1)$, and use that

$$\Bigl(\log \frac{1}{A_j}\Bigr)^\alpha \geq C \cdot 2^{(1+\varepsilon)j}, \qquad j \geq 1,$$

for some $C, \varepsilon > 0$. By (2.26), it remains to check that

$$2^j \log\log \frac{1}{u_k} \leq C\Bigl(2^{(1+\varepsilon)j} + \Bigl(\log \frac{1}{u_k}\Bigr)^\alpha\Bigr).$$

Indeed, if

$$\log\log \frac{1}{u_k} > 2^{\varepsilon j},$$

then

$$C\Bigl(\log \frac{1}{u_k}\Bigr)^\alpha > 2^j \log\log \frac{1}{u_k}.$$

Finally, we fix $\alpha \in (\alpha_0, 1)$ and a Λ_α-Carleson set F. We have

$$\mathbb{T} \setminus F = \sqcup_s L_s^*$$

for some intervals L_s^*. By [3, Theorem B], there exist disjoint intervals $L_{n,s}$ such that

$$F \subset \sqcup_s L_{n,s}, \qquad \sum_s |L_{n,s}| \Lambda_\alpha(|L_{n,s}|) < \frac{1}{n}, \qquad n \geq 1.$$

Then by (2.27),

$$\sum_s |\nu(L_{n,s})| < \frac{c}{n}.$$

Set

$$\mathbb{T} \setminus \sqcup_s L_{n,s} = \sqcup_s L_{n,s}^*.$$

Then

$$\Big|\sum_s \nu(L_{n,s}^*)\Big| < \frac{c}{n}.$$

Since F is Λ_α-Carleson, we have

$$\sum_s |L_s^*| \Lambda_\alpha(|L_s^*|) < \infty,$$

and hence,

$$\sum_s \nu(L_{n,s}^*) \to \sum_s \nu(L_s^*)$$

as $n \to \infty$. Thus,

$$\sum_s \nu(L_s^*) = 0,$$

and hence, $\nu(F) = 0$. Again by Theorem 2.20, f_ν is cyclic in $\mathcal{A}_{\Lambda_\alpha}^{-\infty}$ for $\alpha > \alpha_0$. $\square$.

Chapitre 3

Cyclicity in weighted Bergman type spaces

In this Chapter we study the cyclicity of $S(z) = e^{-\frac{1+z}{1-z}}$ in the spaces $\mathcal{B}^{\infty,0}_{\Lambda,E}$ and $\mathcal{B}^{p}_{\Lambda,E}$.

Given a positive non-increasing continuous function Λ on $(0,1]$ and $E \subset \mathbb{T} = \partial\mathbb{D}$, we denote by $\mathcal{B}^{\infty}_{\Lambda,E}$ the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\Lambda,E,\infty} = \sup_{z\in\mathbb{D}} |f(z)| e^{-\Lambda(\mathrm{dist}(z,E))} < +\infty,$$

and by $\mathcal{B}^{\infty,0}_{\Lambda,E}$ its separable subspace

$$\mathcal{B}^{\infty,0}_{\Lambda,E} = \big\{ f \in \mathcal{B}^{\infty}_{\Lambda,E} : \lim_{\mathrm{dist}(z,E)\to 0} |f(z)| e^{-\Lambda(\mathrm{dist}(z,E))} = 0 \big\}.$$

Analogously, integrating with respect to area measure on the disc, we define the spaces $\mathcal{B}^{p}_{\Lambda,E}$, $1 \le p < \infty$:

$$\mathcal{B}^{p}_{\Lambda,E} = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|^p_{\Lambda,E,p} = \int_{\mathbb{D}} |f(z)|^p e^{-p\Lambda(\mathrm{dist}(z,E))} < +\infty \Big\}.$$

If $E = \mathbb{T}$, we use the notation $\mathcal{B}^{\infty}_{\Lambda}$, $\mathcal{B}^{\infty,0}_{\Lambda}$, $\mathcal{B}^{p}_{\Lambda}$. Let us remark that either $\Lambda(0^+) = +\infty$ or $\mathcal{B}^{\infty}_{\Lambda} = H^\infty$, $\mathcal{B}^{\infty,0}_{\Lambda} = \{0\}$, $\mathcal{B}^{p}_{\Lambda} = \mathcal{B}^{p}_{0}$.

In the first part of this chapter we prove the following theorems :

Theorem 3.1 *Let Λ be a positive non-increasing continuous function on $(0,1]$. Then $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda}$ if and only if Λ satisfies*

$$\int_0 \sqrt{\frac{\Lambda(t)}{t}}\, dt = \infty. \tag{3.1}$$

Theorem 3.2 *Let $1 \le p < \infty$ and let Λ be a positive non-increasing continuous function on $(0,1]$. Then $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^p_\Lambda$ if and only if Λ satisfies*

$$\int_0 \sqrt{\frac{\Lambda(t)}{t}} dt = \infty.$$

3.1 Generalized Phragmén–Lindelöf Principle

We start with the Ahlfors–Carleman estimate of the harmonic measure (see, for example, [30, IX.E1]). Let G be a simply connected domain such that $\infty \in \partial G$. Fix $z_0 \in G$. Given $\rho > 0$ we denote by G_ρ the connected component of the intersection of the disc $\rho\mathbb{D} = \{z : |z| < \rho\}$ and G containing z_0, S_ρ is an arc on $\partial(\rho\mathbb{D}) \cap G$ separating z_0 from one of the unbounded components of $G \setminus \overline{\rho\mathbb{D}}$, $s(\rho)$ is the length of S_ρ. Then

$$\omega(z_0, S_\rho, G_\rho) \le C \exp\Big(-\pi \int_0^\rho \frac{dr}{s(r)}\Big)$$

for an absolute constant ; here $\omega(z_0, \cdot, \Omega)$ is the harmonic measure with respect to $z \in \Omega$ on the boundary of Ω.

Corollary 3.3 *Suppose that G is as above, a function f is analytic in G, continuous up to $\partial G \setminus \{\infty\}$ and satisfies the conditions*

$$|f(z)| \le 1, \qquad z \in \partial G,$$
$$\liminf_{\rho\to\infty} \frac{\log M(\rho)}{\sigma(\rho)} = 0,$$

where

$$M(\rho) = \max_{z\in S_\rho} |f(z)|, \qquad \sigma(\rho) = \exp\Big\{\pi \int_1^\rho \frac{dr}{s(r)}\Big\}.$$

Then $|f(z)| \le 1$, $z \in G$.

Given an increasing differentiable function $\phi : [0, +\infty[\to \mathbb{R}_+$ such that

$$\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty, \tag{3.2}$$

we consider the domain G_ϕ defined by :

$$G_\phi := \big\{x + iy \in \mathbb{C} : |y| \le \phi(x),\ x \ge 0\big\}.$$

Proposition 3.4 *Let f be a function analytic on G_ϕ and continuous up to $\partial G_\phi \setminus \{\infty\}$ such that for some $c > 0$ we have*

$$|f(z)| \le e^{c|z|}, \qquad z \in G_\phi,$$
$$|f(\xi)| \le 1, \qquad \xi \in \partial G_\phi \setminus \{\infty\}.$$

If

$$\int^{\infty} \frac{x\phi'(x)\,dx}{\phi(x)^2} = +\infty, \tag{3.3}$$

then $|f(z)| \le 1$, $z \in G_\phi$.

Proof. In the notations of Corollary 3.3, we have $\log M(\rho) \le c\rho$.

A simple geometric argument shows that if

$$r^2 = x^2 + \phi(x)^2, \tag{3.4}$$

then

$$s(r) = r\Big(\pi - 2\arctan\frac{x}{\phi(x)}\Big).$$

Therefore, if $x = x(r)$ is defined by (3.4), then

$$\frac{\pi}{s(r)} - \frac{1}{r} \ge \frac{x}{2r\phi(x)} \ge \frac{x}{3\phi(x)^2}$$

for large r (we use (3.2)). Moreover,

$$dr = \frac{x + \phi(x)\phi'(x)}{r}\,dx \ge \frac{\phi'(x)\,dx}{2}$$

for large r.

Thus, for some $c > 0$ we have

$$\sigma(\rho) \ge c\rho \cdot \exp\Big[\frac{1}{6}\int_{x(1)}^{x(\rho)} \frac{x\phi'(x)\,dx}{\phi(x)^2}\Big].$$

Therefore, (3.3) implies that

$$\lim_{\rho\to\infty} \frac{\log M(\rho)}{\sigma(\rho)} = 0,$$

and it remains to apply Corollary 3.3. □

By the standard maximum principle, it suffices to require in Proposition 3.4 that $\phi(x)$ is increasing for large x.

Given a positive decreasing differentiable function Λ on $(0, 1]$ such that $\Lambda(0^+) = +\infty$ and

$$t\Lambda(t) = o(1), \qquad t \to 0, \tag{3.5}$$

we consider the domain

$$\Omega_\Lambda = \Big\{ w \in \mathbb{D} : \frac{1-|w|^2}{|1-w|^2} \ge \Lambda(1-|w|^2) \Big\}.$$

Let $F(w) = \frac{1+w}{1-w}$ be a conformal map of the unit disc onto the right half plane, and let $x+iy$ be a point on the boundary of $F(\Omega_\Lambda)$, $y \ge 0$. Then

$$x = \Lambda\Big(\frac{4x}{(x+1)^2+y^2}\Big).$$

By monotonicity of Λ, y is determined uniquely by x, $y = y(x)$, and we have $F(\Omega_\Lambda) = G_y$. Furthermore, we obtain that

$$\frac{4x^2}{(x+1)^2+y^2} = \frac{4x}{(x+1)^2+y^2} \cdot \Lambda\Big(\frac{4x}{(x+1)^2+y^2}\Big) = o(1), \qquad x \to \infty.$$

Hence $x = o(y)$, $x \to \infty$.

Next, for sufficiently small positive t we have

$$\frac{4\Lambda(t)}{(\Lambda(t)+1)^2 + y(\Lambda(t))^2} = t,$$

and, hence, y is differentiable and

$$2\frac{t\Lambda'(t) - \Lambda(t)}{t^2} = \Lambda'(t)\big(\Lambda(t) + 1 + y(\Lambda(t))y'(\Lambda(t))\big).$$

Since $t\Lambda(t) = o(1)$ and $y(\Lambda(t)) = (2+o(1))\sqrt{\Lambda(t)/t}$, $t \to 0$, we obtain

$$\Lambda'(t)y'(\Lambda(t)) = (1+o(1))\frac{t\Lambda'(t)-\Lambda(t)}{t\sqrt{t\Lambda(t)}}, \qquad t \to 0. \tag{3.6}$$

In particular, the function $y(x)$ increases for large x.

Finally, we estimate the integral

$$I = \int^\infty \frac{xy'(x)}{y(x)^2}\,dx.$$

By (3.6) we have

$$I \ge c + \int_0 \frac{t}{5} \cdot \frac{|t\Lambda'(t) - \Lambda(t)|}{t\sqrt{t\Lambda(t)}}\,dt = c + \frac{1}{5}\int_0 \sqrt{\frac{\Lambda(t)}{t}}\,dt + \frac{1}{5}\int_0 |\Lambda'(t)|\sqrt{\frac{t}{\Lambda(t)}}\,dt.$$

Integrating by parts and using (3.5), one can easily verify that the integrals $\int_0 \sqrt{\Lambda(t)/t}\,dt$ and $\int_0 |\Lambda'(t)|\sqrt{t/\Lambda(t)}\,dt$ converge simultaneously. Thus, $I = \infty$ if and only if (3.1) holds.

Corollary 3.5 *Let f be analytic on the domain Ω_Λ and continuous up to $\partial\Omega_\Lambda \setminus \{1\}$, where Λ is a positive decreasing differentiable function on $(0,1]$ satisfying $\Lambda(0^+) = +\infty$ and (3.5). If for some $c > 0$ we have*

(a) $|f(w)| \le e^{c\frac{1-|w|^2}{|1-w|^2}}$, $w \in \Omega_\Lambda$,

(b) $|f(\xi)| \le 1$, $\xi \in \partial\Omega_\Lambda \setminus \{1\}$,

and if

$$\int_0 \sqrt{\frac{\Lambda(t)}{t}}\, dt = \infty,$$

then $|f(z)| \le 1$, $z \in \Omega_\Lambda$.

3.2 Auxiliary estimates

Given $\lambda \in \mathbb{D}$, we define the Privalov shadow I_λ,

$$I_\lambda = \Big\{ e^{i\theta} : |\arg(\lambda e^{-i\theta})| < \frac{1-|\lambda|}{2} \Big\}$$

(see Figure 3.1), which is an arc of the circle $\mathbb{T}$ centered at $\frac{\lambda}{|\lambda|}$ (the radial projection of λ onto the unit circle $\mathbb{T}$) of length $|I_\lambda| = 1 - |\lambda|$. Furthermore, we consider an auxiliary function

$$f_\lambda(z) = \exp\Big\{ c_\lambda \frac{1-|\lambda|^2}{|1-\lambda|^2} \int_{I_\lambda} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta \Big\},$$

where

$$c_\lambda^{-1} = \int_{I_\lambda} \frac{1-|\lambda|^2}{|e^{i\theta}-\lambda|^2}\, d\theta.$$

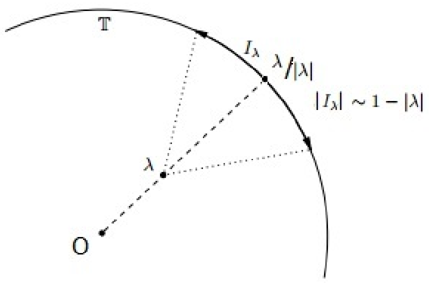

FIGURE 3.1 – Privalov shadow.

Then

$$|f_\lambda(\lambda) S(\lambda)| = 1,$$

where $S(z) = \exp[-(1+z)/(1-z)]$. Next, a geometric argument shows that

$$c_\lambda^{-1} \geq \frac{1-|\lambda|^2}{5/4(1-|\lambda|)^2} \cdot (1-|\lambda|) \geq \frac{4}{5},$$

and we have

$$\sup_{\mathbb{D}} |f_\lambda| \leq \exp\Big(\frac{5\pi}{2} \cdot \frac{1-|\lambda|^2}{|1-\lambda|^2}\Big).$$

Given $a > 0$, $A > 1$, we consider the domain $\Gamma_{\Lambda,\mathbb{T}}(a, A)$ defined by

$$\Gamma_{\Lambda,\mathbb{T}}(a, A) = \Big\{\lambda \in \mathbb{D} : \frac{1-|\lambda|^2}{|1-\lambda|^2} \leq a\Lambda(A(1-|\lambda|))\Big\}.$$

Now for some a, A we establish the following estimate :

Lemma 3.6

$$\sup_{\lambda \in \Gamma_{\Lambda,\mathbb{T}}(a,A)} \|f_\lambda S\|_\Lambda < \infty. \tag{3.7}$$

Proof. We set

$$H_\lambda(z) = c_\lambda \frac{1-|\lambda|^2}{|1-\lambda|^2} \int_{I_\lambda} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\, d\theta - \frac{1-|z|^2}{|1-z|^2} - \Lambda(1-|z|),$$

and obtain

$$|f_\lambda(z)S(z)|e^{-\Lambda(1-|z|)} = e^{H_\lambda(z)}.$$

Thus, it remains to verify that

$$\sup_{\lambda \in \Gamma_{\Lambda,\mathbb{T}}(a,A),\, z \in \mathbb{D}} H_\lambda(z) < \infty.$$

Case 1 : If $A(1-|\lambda|) \geq 1-|z|$, then

$$\begin{aligned} H_\lambda(z) &\leq 2\pi c_\lambda \frac{1-|\lambda|^2}{|1-\lambda|^2} - \Lambda(1-|z|) \\ &\leq \frac{5\pi a}{2}\Lambda(A(1-|\lambda|)) - \Lambda(A(1-|\lambda|)) = \Big(\frac{5\pi a}{2} - 1\Big)\Lambda(A(1-|\lambda|)). \end{aligned}$$

Therefore, for $a \leq \frac{2}{5\pi}$ we obtain that $H_\lambda(z) \leq 0$.

Case 2 : If $|1-z| \geq 6|1-\lambda|$, then for every $e^{i\theta} \in I(\lambda)$ we have

$$\begin{aligned} |z-e^{i\theta}| &\geq |1-z| - |1-\lambda| - |\lambda - e^{i\theta}| \geq |1-z| - |1-\lambda| - 2(1-|\lambda|) \\ &\geq |1-z| - 3|1-\lambda| \geq \frac{1}{2}|1-z|. \end{aligned}$$

Therefore,

$$H_\lambda(z) \le 5\frac{1-|\lambda|^2}{|1-\lambda|^2}\frac{1-|z|^2}{|1-z|^2}(1-|\lambda|) - \frac{1-|z|^2}{|1-z|^2} \le \Big[5a(1-|\lambda|)\Lambda(A(1-|\lambda|)) - 1\Big]\frac{1-|z|^2}{|1-z|^2}.$$

Since $t\Lambda(t) = o(1)$, $t \to 0$, we obtain that $H_\lambda(z)$ is uniformly bounded.

Case 3 : If $A(1-|\lambda|) < 1-|z|$ and $|1-z| < 6|1-\lambda|$, then

$$\frac{1-|z|^2}{|1-z|^2} \ge \frac{A}{72}\cdot\frac{1-|\lambda|^2}{|1-\lambda|^2},$$

and hence,

$$H_\lambda(z) \le \frac{180\pi}{A}\cdot\frac{1-|z|^2}{|1-z|^2} - \frac{1-|z|^2}{|1-z|^2} \le 0,$$

for $A \ge 1000$. □

Now, let E be an arbitrary compact subset of $\mathbb{T}$. For some $a > 0$, $A > 1$, we consider the domain $\Gamma_{\Lambda,E}(a,A)$ defined by

$$\Gamma_{\Lambda,E}(a,A) = \Big\{\lambda \in \mathbb{D} : \frac{1-|\lambda|^2}{|1-\lambda|^2} \le a\Lambda(A\,\mathrm{dist}(\lambda,E))\Big\},$$

and obtain the estimate

Lemma 3.7

$$\sup_{\lambda\in\Gamma_{\Lambda,E}(a,A)} \|f_\lambda S\|_\Lambda < \infty.$$

Proof. We just need to verify that

$$\sup_{\lambda\in\Gamma_{\Lambda,E}(a,A),\, z\in\mathbb{D}} H_\lambda(z) < \infty.$$

In the cases (1) $A\,\mathrm{dist}(\lambda,E) \ge \mathrm{dist}(z,E)$ and (2) $|1-z| \ge 6|1-\lambda|$ we use the same argument as in the proof of Lemma 3.6. In the case (3) we have $A\,\mathrm{dist}(\lambda,E) < \mathrm{dist}(z,E)$ and $|1-z| < 6|1-\lambda|$. Take $e^{i\eta} \in E$ such that $A|\lambda - e^{i\eta}| \le |z - e^{i\eta}|$ and use that for $e^{i\theta} \in I_\lambda$ we have

$$1-|\lambda| \le |\lambda - e^{i\theta}| \le 2|\lambda - e^{i\eta}|.$$

Then,

$$|z-e^{i\theta}| \ge |z-e^{i\eta}| - |\lambda-e^{i\theta}| - |\lambda-e^{i\eta}| \ge (A-3)|\lambda-e^{i\eta}| \ge \frac{A-3}{2}(1-|\lambda|), \qquad e^{i\theta}\in E.$$

Therefore,

$$H_\lambda(z) \le \frac{5}{4}\frac{1-|\lambda|^2}{|1-\lambda|^2}\frac{4}{(A-3)^2}\frac{1-|z|^2}{|1-z|^2}(1-|\lambda|) - \frac{1-|z|^2}{|1-z|^2} \le \Big(\frac{360}{(A-3)^2} - 1\Big)\frac{1-|z|^2}{|1-z|^2} \le 0$$

for $A \ge 100$. □

3.3 Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1. Suppose that Λ is non-increasing and satisfies (3.1). Then $\Lambda(0^+) = \infty$, and slightly decreasing, if necessary, Λ, we can assume that Λ is decreasing.

If $\int_0 \Lambda(t)\,dt = \infty$, then the result follows from [17, Theorem 2]. Therefore, from now on we assume that $\int_0 \Lambda(t)\,dt < \infty$. Since Λ decreases, we have

$$\Lambda(t) \le \frac{1}{t}\int_0^t \Lambda(s)\,ds = o\Big(\frac{1}{t}\Big), \qquad t \to 0,$$

and, hence, $t\Lambda(t) = o(1)$, $t \to 0$. Finally, smoothing, if necessary, Λ, we can assume that Λ is differentiable.

We denote by π the canonical projection of $\mathcal{B}_\Lambda^{\infty,0}$ onto $\mathcal{B}_\Lambda^{\infty,0}/[S]$, and by $\alpha : z \mapsto z$ the identity map.

Suppose that $1 \notin [S]$. Next, we estimate $\|(\lambda - \pi(\alpha))^{-1}\pi(1)\|$. Given $\lambda \in \mathbb{D}$ and an analytic function f, we define the following function :

$$L_\lambda(f)(z) = \begin{cases} \dfrac{f(z) - f(\lambda)}{z - \lambda} & if \quad z \ne \lambda \\ \\ f'(z) & if \quad z = \lambda\,. \end{cases}$$

For bounded functions f we have

$$f(\lambda)S(\lambda)\pi(1) = (\lambda - \pi(\alpha))\pi(L_\lambda(fS)). \tag{3.8}$$

In particular,

$$S(\lambda)\pi(1) = (\lambda - \pi(\alpha))\pi(L_\lambda(S)), \qquad \lambda \in \mathbb{C} \setminus \{1\},$$

and hence, the function $\lambda \mapsto (\lambda - \pi(\alpha))^{-1}\pi(1)$ is well defined and analytic in $\mathbb{C} \setminus \{1\}$.

Next we use that the function $Q(z) = \log|f_\lambda(z)S(z)|$ is the Poisson integral

$$Q(z) = \mathcal{P}(z,\theta) * d\mu(\theta)$$

of a finite (signed) measure

$$d\mu(\theta) = 2\pi c_\lambda \frac{1 - |\lambda|^2}{|1 - \lambda|^2}\chi_{I_\lambda}(\theta)\,d\theta - \delta_1$$

with mass

$$|\mu|(\mathbb{T}) = 2\pi c_\lambda \frac{(1 - |\lambda|^2)(1 - |\lambda|)}{|1 - \lambda|^2} + 1$$

uniformly bounded in λ. Since

$$\sup_{z\in\mathbb{D},\,\theta\in\mathbb{T}} (1 - |z|)^2 \|\nabla P(z,\theta)\| < \infty,$$

for an absolute constant c we obtain that

$$\|\nabla Q(z)\| \le \frac{c}{(1-|z|)^2}.$$

Since $Q(\lambda)=0$, we have $|f_\lambda(z)S(z)| \le c_1$ at the boundary of the disc $D_\lambda = \{z : |z-\lambda| < (1-|\lambda|)^2/2\}$, and hence, by the maximum principle,

$$\Big|\frac{f_\lambda(z)S(z) - f_\lambda(\lambda)S(\lambda)}{z-\lambda}\Big| \le \frac{2(c_1+1)}{(1-|\lambda|)^2}, \qquad z \in D_\lambda.$$

Therefore,

$$\begin{aligned}
\|L_\lambda(f_\lambda S)\|_\Lambda &= \sup_{|z|<1}\Big|\frac{f_\lambda(z)S(z) - f_\lambda(\lambda)S(\lambda)}{z-\lambda}\Big| e^{-\Lambda(1-|z|)} \\
&\le \sup_{|z-\lambda|\le(1-|\lambda|)^2/2}\Big|\frac{f_\lambda(z)S(z) - f_\lambda(\lambda)S(\lambda)}{z-\lambda}\Big| e^{-\Lambda(1-|z|)} \\
&+ \sup_{|z-\lambda|>(1-|\lambda|)^2/2}\Big|\frac{f_\lambda(z)S(z) - f_\lambda(\lambda)S(\lambda)}{z-\lambda}\Big| e^{-\Lambda(1-|z|)} \\
&\le \frac{2}{(1-|\lambda|)^2}\big(\|f_\lambda S\|_\Lambda + c_1 + 2\big).
\end{aligned}$$

By (3.8), for every $\lambda \in \mathbb{D}$ we have

$$\|(\lambda - \pi(\alpha))^{-1}\pi(1)\|_\Lambda \le \frac{1}{|f_\lambda(\lambda)S(\lambda)|}\|L_\lambda(f_\lambda S)\|_\Lambda \le \frac{2}{(1-|\lambda|)^2}\big(\|f_\lambda S\|_\Lambda + c_1 + 2\big). \tag{3.9}$$

In the same way, by (3.8), for every $\lambda \in \mathbb{D}$ we have

$$\|(\lambda - \pi(\alpha))^{-1}\pi(1)\|_\Lambda \le \frac{1}{|S(\lambda)|}\|\pi(L_\lambda(S))\|_\Lambda \le \frac{2}{1-|\lambda|}\exp\frac{1-|\lambda|^2}{|1-\lambda|^2}. \tag{3.10}$$

Furthermore, for $|\lambda|>1$ we have

$$\|(\lambda - \pi(\alpha))^{-1}\pi(1)\|_\Lambda \le \frac{1}{|S(\lambda)|}\|\pi(L_\lambda(S))\|_\Lambda \le \frac{2}{|\lambda|-1}. \tag{3.11}$$

Let us fix a and A such that (3.7) holds. For $\lambda \in \partial\Gamma_\Lambda(a,A)\setminus\mathbb{T}$ we have $1-|\lambda|^2 = a\Lambda(A(1-|\lambda|))|1-\lambda|^2$, and hence,

$$\frac{1}{1-|\lambda|} \le \frac{c}{|1-\lambda|^2}, \qquad \lambda \in \partial\Gamma_\Lambda(a,A)\setminus\mathbb{T}. \tag{3.12}$$

By Lemma 3.6 and (3.9), we have

$$\sup_{\lambda\in\partial\Gamma_\Lambda(a,A)} \|(\lambda-1)^4(\lambda - \pi(\alpha))^{-1}\pi(1)\| < \infty. \tag{3.13}$$

Applying (3.10) and Corollary 3.5 (to Λ_1 defined by $\Lambda_1(1-r^2) = a\Lambda(A(1-r))$), we obtain that the function $(\lambda-1)^4(\lambda-\pi(\alpha))^{-1}\pi(1)$ is bounded on $\mathbb{D}\setminus\Gamma_\Lambda(a,A) = \operatorname{Int}\Omega_{\Lambda_1}$.

By (3.12), for some $c>0$ we have

$$\operatorname{dist}(\zeta, \partial\Gamma_\Lambda(a,A)\setminus\mathbb{T}) \geq c|1-\zeta|^2, \qquad \zeta\in\mathbb{T}.$$

Applying Levinson's log-log theorem (see, for example, [30, VII D7]) or, rather, its polynomial growth version to the function $(\lambda-\pi(\alpha))^{-1}\pi(1)$ and using estimates (3.13) and (3.11) we obtain that

$$\|(\lambda-\pi(\alpha))^{-1}\pi(1)\| \leq \frac{C}{|\lambda-1|^4}, \qquad \lambda\in 2\mathbb{D}\setminus(\{1\}\cup\mathbb{D}\setminus\Gamma_\Lambda(a,A)). \tag{3.14}$$

By (3.13) and (3.14), the function $\lambda\mapsto(\lambda-1)^4(\lambda-\pi(\alpha))^{-1}\pi(1)$ is bounded on $2\mathbb{D}\setminus\{1\}$.

Now, pick an arbitrary functional $\phi\perp[S]$, and consider the function

$$\Phi(\lambda) = (\lambda-1)\langle(\lambda-\pi(\alpha))^{-1}\pi(1),\phi\rangle.$$

The function Φ is analytic on $\mathbb{C}\setminus\{1\}$ and has a pole of order at most 3 at the point 1. By (3.11), we have

$$\limsup_{\varepsilon\to0^+}|\Phi(1+\varepsilon)|<\infty,$$

and hence, Φ is an entire function. Again by (3.11), Φ is bounded and hence is a constant function. Furthermore,

$$\begin{aligned}\Phi(\lambda) = \langle(\lambda-\pi(\alpha))^{-1}\pi(\lambda-1),\phi\rangle &= \langle(\lambda-\pi(\alpha))^{-1}\pi(\lambda-\alpha),\phi\rangle + \langle(\lambda-\pi(\alpha))^{-1}\pi(\alpha-1),\phi\rangle \\ &= \langle\pi(1),\phi\rangle + \langle(\lambda-\pi(\alpha))^{-1}\pi(\alpha-1),\phi\rangle.\end{aligned}$$

Arguing as in the proof of (3.11), we see that

$$\lim_{|\lambda|\to\infty}(\lambda-\pi(\alpha))^{-1}\pi(\alpha-1)=0,$$

and hence, $\langle\alpha-1,\phi\rangle=0$. Since ϕ is an arbitrary functional vanishing on $[S]$, we get $\alpha-1\in[S]$, $\alpha^n-1\in[S]$, $n\geq1$, and hence, $1\in[S]$. This contradiction shows that $[S]=\mathcal{B}^{\infty,0}_\Lambda$.

In the opposite direction, suppose that S is cyclic in $\mathcal{B}^{\infty,0}_\Lambda$. We replace Λ by Λ^*,

$$\Lambda^*(t) = \Lambda(t) + \log\frac{1}{t},$$

and remark that S is also cyclic in $\mathcal{B}^{\infty,0}_{\Lambda^*}$. Now we can apply the result by Nikolski [39, Theorem 1a, Section 2.6] to conclude that Λ^* satisfies (3.1). □

Proof of Theorem 3.2. It suffices to remark that

$$\mathcal{B}^{\infty,0}_\Lambda\subset\mathcal{B}^p_\Lambda\subset\mathcal{B}^{\infty,0}_{\Lambda^*},$$

where

$$\Lambda^*(t) = \Lambda(t/2) + \frac{2}{p}\log\frac{1}{t},$$

and to apply Theorem 3.1. We use here the subharmonicity of $|f|^p$ and the fact that Λ and Λ^* satisfy (3.1) simultaneously. □

Remark 3.8 *The same method works for the spaces $\mathcal{B}^p_{\Lambda,E}$, where E is a closed arc of the unit circle. More precisely we have the following result :*

Let E be a non-trivial closed arc of the unit circle such $1 \in E$, let $1 \le p < \infty$, and let Λ be a positive non-increasing continuous function on $(0, 1]$. Then $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^p_{\Lambda,E}$ if and only if Λ satisfies (3.1).

Sketch of the proof. For the necessity part it suffices to use that $\mathcal{B}^p_{\Lambda,E} \subset \mathcal{B}^p_{\Lambda}$. In the opposite direction, suppose that Λ satisfies (3.1) and let $E = \overline{(e^{ia}, e^{ib})}$. If $1 \in (e^{ia}, e^{ib})$ then the same proof works. In the case $e^{ia} = 1$ a slight modification is needed : we replace Ω_Λ by

$$\widetilde{\Omega}_\Lambda = \Big\{w \in \mathbb{D} : \operatorname{Im} w \ge 0,\ \frac{1-|w|^2}{|1-w|^2} \ge a\Lambda(A(1-|w|^2))\Big\} \\ \cup \Big\{w \in \mathbb{D} : \operatorname{Im} w \le 0,\ \frac{1-|w|^2}{|1-w|^2} \ge a_1\Lambda(A_1|1-w|)\Big\}$$

for some a, a_1, A, A_1. □

3.4 An auxiliary domain for general E

Let E be a compact subset of $\mathbb{T}$, $1 \in E$. Given a positive decreasing C^1 smooth function Λ on $(0, 1)$ such that $\Lambda(1) < 1/10$, and

$$\lim_{t\to 0}\Lambda(t) = \infty, \qquad \lim_{t\to 0} t\Lambda(t) = 0, \qquad t|\Lambda'(t)| = O(\Lambda(t)), \quad t \to 0, \tag{3.15}$$

we consider the domain

$$\Omega_{\Lambda,E} = \{(1-s)e^{i\theta} : s \ge \theta^2\Lambda(s + \operatorname{dist}(e^{i\theta}, E)),\ \theta \in (-\pi, \pi]\}.$$

Clearly, $\Omega_{\Lambda,E}$ is star-shaped with respect to the origin,

$$\partial\Omega_{\Lambda,E} = \{(1-\gamma(\theta))e^{i\theta},\ \theta \in (-\pi, \pi]\}$$

with

$$\gamma(\theta) = \theta^2\Lambda(\gamma(\theta) + \operatorname{dist}(e^{i\theta}, E)). \tag{3.16}$$

Next,

$$\inf_{\theta\in(-\pi,\pi]\setminus\{0\}} \frac{\gamma(\theta)}{\theta^2} > 0, \tag{3.17}$$

$\lim_{\theta\to 0}\gamma(\theta) = 0$, and

$$\frac{\gamma(\theta)}{\theta} = [\gamma(\theta)\Lambda(\gamma(\theta) + \operatorname{dist}(e^{i\theta}, E))]^{1/2} = o(1), \qquad \theta \to 0.$$

Furthermore, the derivative $h(\theta)$ of $\operatorname{dist}(e^{i\theta}, E)$ is equal to ± 1 for a.e. $e^{i\theta}$ on $\mathbb{T}\setminus E$ and to 0 for a.e. $e^{i\theta}$ on E. Therefore,

$$\gamma'(\theta) = 2\theta\Lambda(\gamma(\theta) + \operatorname{dist}(e^{i\theta}, E)) - \theta^2|\Lambda'(\gamma(\theta) + \operatorname{dist}(e^{i\theta}, E))|(\gamma'(\theta) + h(\theta))$$

for a.e. $e^{i\theta} \in \mathbb{T}$, and hence, by (3.15),

$$|\gamma'(\theta)| = O(1), \qquad \text{a.e. } \theta \to 0. \tag{3.18}$$

Now we set

$$(1-\gamma(\theta))e^{i\theta} = 1 - \frac{e^{i\phi}}{R},$$

with $\phi \in [-\pi/2, \pi/2]$. Then

$$\begin{aligned} R &\asymp \frac{1}{\theta}, \qquad \frac{\pi}{2} - |\phi| \asymp \frac{\gamma(\theta)}{\theta}, \\ \frac{dR}{d\theta} &= (1+o(1))R^2, \qquad \theta \to 0. \end{aligned}$$

By analogy with Proposition 3.4 and Corollary 3.5 we have

Proposition 3.9 *Given a continuous function $\phi : \mathbb{R}_+ \to (0, \pi/2)$ let*

$$G_\phi := \big\{Re^{i\theta} : |\theta| < \frac{\pi}{2} - \phi(R),\ R > 0\big\}.$$

Let f be a function analytic on G_ϕ and continuous up to $\partial G_\phi \setminus \{\infty\}$ such that for some $c > 0$ we have

$$\begin{aligned} |f(z)| &\le e^{c|z|}, \qquad z \in G_\phi, \\ |f(\xi)| &\le 1, \qquad \xi \in \partial G_\phi \setminus \{\infty\}. \end{aligned}$$

If

$$\int^\infty \frac{\phi(R)\,dR}{R} = +\infty, \tag{3.19}$$

then $|f(z)| \le 1$, $z \in G_\phi$.

Corollary 3.10 *Let f be analytic on the domain $\Omega_{\Lambda,E}$ and continuous up to $\partial\Omega_{\Lambda,E} \setminus \{1\}$, where Λ is a positive decreasing differentiable function on $(0,1]$ satisfying* (3.15). *If for some $c > 0$ we have*

(a) $|f(w)| \le e^{c\frac{1-|w|^2}{|1-w|^2}}$, $w \in \Omega_{\Lambda,E}$,

(b) $|f(\xi)| \le 1$, $\xi \in \partial\Omega_{\Lambda,E} \setminus \{1\}$,

and if

$$\int_0 \frac{\gamma(\theta)}{\theta^2}\, d\theta = +\infty,$$

then $|f(z)| \le 1$, $z \in \Omega_{\Lambda,E}$.

Later on, we need the following result.

Proposition 3.11 *Let Λ be a positive decreasing differentiable function on $(0,1]$ satisfying* (3.15), *and let*

$$\int_0 \frac{\gamma(\theta)}{\theta^2}\, d\theta < +\infty. \tag{3.20}$$

There exists an outer function F such that

$$|F(w)| > e^{\frac{1-|w|^2}{|1-w|^2} + \Lambda(\mathrm{dist}(w,E))}, \qquad w \in \partial\Omega_{\Lambda,E} \setminus \{1\}. \tag{3.21}$$

Proof. By (3.20), we can set

$$\log|F(e^{i\theta})| = A\frac{\gamma(\theta)}{\theta^2}$$

for some A to be chosen later.

Given $w = (1-\gamma(\theta))e^{i\theta} \in \partial\Omega_{\Lambda,E}$ and $\beta > 0$ we have

$$\log|F(w)| \ge \beta_1 \min_{|\psi-\theta|<\beta\gamma(\theta)} \frac{\gamma(\psi)}{\psi^2}$$

with $\beta_1 = \beta_1(\beta) > 0$. Furthermore, by (3.18), for some $\beta > 0$ we have

$$\min_{|\psi-\theta|<\beta\gamma(\theta)} \frac{\gamma(\psi)}{\psi^2} \ge \frac{\gamma(\theta)}{2\theta^2}.$$

Now, (3.21) follows for sufficiently large A. □

3.5 General E

Let Λ be a positive decreasing differentiable function on $(0,1]$ satisfying (3.15), and let E be a compact subset of $\mathbb{T}$, $1 \in E$. We define γ by (3.16).

Theorem 3.12 *The function $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$ if and only if the integral*

$$\int \frac{\gamma(\theta)}{\theta^2}\, d\theta \tag{3.22}$$

diverges at 0.

Proof. If the integral diverges, we use the same method as in the proof of Theorem 3.1 ; we use Corollary 3.10 instead of Corollary 3.5 and Lemma 3.7 instead of Lemma 3.6. The estimate (3.12) is replaced by (3.17).

In the opposite direction, if the integral converges, we use Proposition 3.11 and prove that S is not cyclic by the Keldysh method (see [25], [39, Section 2.8.2]). $\square$

From now on we assume that

$$\Lambda(t) = \frac{1}{tw(t)^2}, \tag{3.23}$$

where w is a positive decreasing C^1 smooth function on $(0,1)$, $\lim_{t\to 0} w(t) = +\infty$, $w(t^2) \asymp w(t)$, $|w'(t)| = O(w(t)/t)$, $t \to 0$. Such Λ satisfy (3.15). Typical w are $\log^p(1/t)$, $p > 0$.

Remark 3.13 *For such Λ, the theorem of Nikolski [39] implies that if $\int_0 \frac{dt}{|t|w(|t|)} < \infty$, then S is not cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$; the theorem of Gevorkyan–Shamoyan [20] implies that if $\int_0 \frac{dt}{|t|w(|t|)^2} = \infty$, then S is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$.*

Let I_n be the arcs complementary to E, $I_n = (e^{ia_n}, e^{ib_n})$ or $I_n = (e^{-ib_n}, e^{-ia_n})$, $0 < a_n < b_n$. We divide the family of all such arcs into three groups :

the short intervals : $\mathcal{I}_1 = \left\{ I_n : 1 - \frac{a_n}{b_n} < \frac{2}{w(b_n)} \right\}$,

the intermediate intervals : $\mathcal{I}_2 = \left\{ I_n : \frac{2}{w(b_n)} \le 1 - \frac{a_n}{b_n} < \frac{1}{2} \right\}$,

and the long intervals : $\mathcal{I}_3 = \left\{ I_n : \frac{a_n}{b_n} \le \frac{1}{2} \right\}$.

Theorem 3.14 *Let Λ be defined by* (3.23) *with w satisfying the above conditions. The function $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$ if and only if*

$$\int_{e^{it}\in E\cup\bigcup_{I_n\in\mathcal{I}_1} I_n} \frac{dt}{|t|w(|t|)} + \sum_{I_n\in\mathcal{I}_2} \frac{1}{w(b_n)^2} \log\Big[\Big(1 - \frac{a_n}{b_n}\Big) w(b_n)\Big] + \sum_{I_n\in\mathcal{I}_3} \frac{\log w(b_n)}{w(b_n)^2} + \int_{e^{it}\in\bigcup_{I_n\in\mathcal{I}_3} I_n} \frac{dt}{|t|w^2(|t|)} = +\infty. \tag{3.24}$$

Remark 3.15 *One can easily verify that condition* (3.24) *is equivalent to the divergence of at least one of the following three expressions :*

$$\int_{e^{it}\in E}\frac{dt}{|t|w(|t|)},\quad \int_0 \frac{dt}{|t|w^2(|t|)},\quad \sum_n \frac{1}{w(b_n)^2}\log\Big[1+\Big(1-\frac{a_n}{b_n}\Big)w(b_n)\Big],$$

where the sums runs by all the arcs I_n complementary to E.

Proof. By Theorem 3.12, we just need to study the convergence of the integral

$$\int \frac{\gamma(t)}{t^2}\,dt$$

at 0.

(a). For $e^{it}\in E$, $t>0$, we have

$$\frac{\gamma(t)}{t^2}=\Lambda(\gamma(t))=\frac{1}{\gamma(t)w(\gamma(t))^2},$$

and hence,

$$\begin{aligned}\gamma(t)w(\gamma(t))&=t,\\ \gamma(t)&\asymp\frac{t}{w(t)},\\ \frac{\gamma(t)}{t^2}&\asymp\frac{1}{tw(t)}.\end{aligned}\tag{3.25}$$

Here we use that under our conditions on w, the function inverse to $t\mapsto tw(t)$ is equivalent to $t\mapsto t/w(t)$.

(b). Let $e^{it}\in I=\{e^{is}:0<a<s<b\}\in\mathcal{I}_1$. (The case $b<t<a<0$ is treated analogously.) Then

$$\frac{\gamma(t)}{t^2}=\Lambda(\gamma(t)+\operatorname{dist}(e^{it},E))\le\Lambda(\gamma(t))$$

and

$$\operatorname{dist}(e^{it},E)<|b-a|<\frac{2b}{w(b)}.$$

Hence,

$$\gamma(t)\lesssim\frac{t}{w(\gamma(t))}\lesssim\frac{t}{w(t)},$$

and

$$\begin{aligned}\Lambda(\gamma(t)+\operatorname{dist}(e^{it},E))&\gtrsim\Lambda(\gamma(t)),\\ \gamma(t)&\gtrsim\frac{t}{w(\gamma(t))}\gtrsim\frac{t}{w(t)}.\end{aligned}$$

Finally,

$$\frac{\gamma(t)}{t^2} \asymp \frac{1}{tw(t)}. \tag{3.26}$$

(c). Let $e^{it} \in I = \{e^{is} : 0 < a < s < b\} \in \mathcal{I}_2$. We have

$$\frac{\gamma(t)}{t^2} \asymp \frac{1}{(\gamma(t) + \operatorname{dist}(t, \{a, b\}))w(b)^2},$$

and hence,

$$\gamma(t)(\gamma(t) + \operatorname{dist}(t, \{a, b\})) \asymp \frac{b^2}{w(b)^2}.$$

Therefore, for some $c > 0$ and for

$$b - \frac{b}{cw(b)} < t < b$$

we have

$$\gamma(t) \asymp \frac{b}{w(b)},$$

$$\int_{b-\frac{b}{cw(b)}}^{b} \frac{\gamma(t)}{t^2}\, dt \asymp \frac{1}{w(b)^2},$$

and for

$$\frac{a+b}{2} < t < b - \frac{b}{cw(b)}$$

we have

$$\gamma(t) \asymp \frac{b^2}{w(b)^2 \operatorname{dist}(t, \{a, b\})},$$

$$\int_{(a+b)/2}^{b-\frac{b}{cw(b)}} \frac{\gamma(t)}{t^2}\, dt \asymp \frac{1}{w(b)^2} \log\Big[c\Big(1 - \frac{a}{b}\Big)w(b)\Big].$$

The integral from a to $(a + b)/2$ is estimated in an analogous way, and we get

$$\int_a^b \frac{\gamma(t)}{t^2} \asymp \frac{1}{w(b)^2} \log\Big[\Big(1 - \frac{a}{b}\Big)w(b)\Big]. \tag{3.27}$$

(d). Let $I = \{e^{is} : 0 < a < s < b\} \in \mathcal{I}_3$. As in part (c), the integral

$$\int_{(a+b)/2}^{b} \frac{\gamma(t)}{t^2}$$

is equivalent to

$$\frac{\log w(b)}{w(b)^2}.$$

Next,

$$\int_a^{a+\frac{a}{w(a)}} \frac{\gamma(t)}{t^2} \asymp \frac{1}{w(a)^2}.$$

For $t \in (a+\frac{a}{w(a)}, \frac{a+b}{2})$ we have

$$\frac{\gamma(t)}{t^2} = \Lambda(\gamma(t)+(t-a)) \le \Lambda(\gamma(t)),$$

and

$$\gamma(t) \le \frac{t}{w(\gamma(t))} \lesssim \frac{t}{w(t)}.$$

Therefore,

$$\frac{\gamma(t)}{t^2} \asymp \Lambda(t-a),$$

and

$$\int_{a+\frac{a}{w(a)}}^{(a+b)/2} \frac{\gamma(t)}{t^2}\,dt \asymp \int_{\frac{a}{w(a)}}^{(b-a)/2} \frac{dt}{tw(t)^2} \asymp \int_a^b \frac{dt}{tw(t)^2} + \frac{\log w(a)}{w(a)^2}.$$

Thus,

$$\int_a^b \frac{\gamma(t)}{t^2}\,dt \asymp \int_a^b \frac{dt}{tw(t)^2} + \frac{\log w(b)}{w(b)^2}. \tag{3.28}$$

The theorem follows from (3.25)–(3.28). □

Corollary 3.16 *Let Λ be as in the formulation of Theorem 3.14. This theorem yields immediately that if $E = \{\exp(i\cdot 2^{-n})\}_{n\ge 1} \cup \{1\}$, then S is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$ if and only if*

$$\sum_{n\ge 1} \frac{\log w(2^{-n})}{w(2^{-n})^2} = +\infty\,;$$

if $E = \{\exp(i\cdot 2^{-2^n})\}_{n\ge 1} \cup \{1\}$, then S is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda,E}$ if and only if

$$\int_0 \frac{dt}{tw(t)^2} = +\infty,$$

and we return to the Gevorkyan–Shamoyan condition valid for $E = \{1\}$.

Next we give two more applications of the general criterion (3.24).

Let us introduce a condition

$$\int_0 \frac{\Lambda(t)^{1-\beta}}{t^\beta}\,dt = +\infty, \qquad 0 \le \beta \le \frac{1}{2} \tag{C_β}$$

interpolating between that by Nikolski ($\int_0 \sqrt{\Lambda(t)/t}\,dt = +\infty$, $\beta = 1/2$) and that by Gevorkyan–Shamoyan ($\int_0 \Lambda(t)\,dt = +\infty$, $\beta = 0$).

For

$$\Lambda_\alpha(t) = \frac{1}{t\log^\alpha(1/t)}$$

we have

$$\Lambda_\alpha \in (C_\beta) \iff \alpha(1-\beta) \le 1.$$

Theorem 3.17 *Let* $0 \le \beta \le 1/2$, $a_n = \exp(-n^{1-\beta})$, $n \ge 1$, $E_\beta = \{e^{ia_n}\}_{n\ge1} \cup \{1\}$. *The function* $S(z) = e^{-\frac{1+z}{1-z}}$ *is cyclic in* $\mathcal{B}^{\infty,0}_{\Lambda_\alpha,E_\beta}$ *if and only if* $\Lambda_\alpha \in (C_\beta)$.

Proof. We have

$$1 - \frac{a_{n+1}}{a_n} = 1 - \exp\bigl[n^{1-\beta} - (n+1)^{1-\beta}\bigr] \asymp n^{-\beta}.$$

Consider three cases.

(a). $\alpha(1-\beta) > 1$. Then all the arcs $\{e^{it}\}_{a_{n+1}<t<a_n}$ are intermediate ones, and we need only to verify that

$$\sum_{n\ge1} \frac{1}{\log^\alpha(\exp(n^{1-\beta}))} \log\Bigl[n^{-\beta}\log^{\alpha/2}(\exp(n^{1-\beta}))\Bigr] \asymp \sum_{n\ge1} \frac{\log(n^{-\beta+\alpha(1-\beta)/2})}{n^{\alpha(1-\beta)}} < +\infty.$$

(b). $2\beta < \alpha(1-\beta) \le 1$. Again all the arcs $\{e^{it}\}_{a_{n+1}<t<a_n}$ are intermediate ones, and

$$\sum_{n\ge1} \frac{1}{\log^\alpha(\exp(n^{1-\beta}))} \log\Bigl[n^{-\beta}\log^{\alpha/2}(\exp(n^{1-\beta}))\Bigr] \asymp \sum_{n\ge1} \frac{\log(n^{-\beta+\alpha(1-\beta)/2})}{n^{\alpha(1-\beta)}} = +\infty.$$

(c). $\alpha(1-\beta) \le 2\beta \le 1$. In this case we can assume that all the arcs are short ones, and we have

$$\int_0 \frac{dt}{t\log^{\alpha/2}(1/t)} = +\infty.$$

Together, (a), (b), and (c) prove the assertion of the theorem. □

Finally, we deal with the Cantor ternary set F. Let $F_0 = [0,1]$. On step $n \ge 0$, F_n consists of 2^n intervals $I_j = [a_j, b_j]$. We divide each of them into three equal subintervals

$$I_j = \bigl[a_j, \frac{2a_j+b_j}{3}\bigr] \cup \bigl[\frac{2a_j+b_j}{3}, \frac{a_j+2b_j}{3}\bigr] \cup \bigl[\frac{a_j+2b_j}{3}, b_j\bigr]$$

and set

$$F_{n+1} = \bigcup_j I_j^1 \cup \bigcup_j I_j^3.$$

We define $F = \cap_{n\ge1} F_n$. Denote by κ the Hausdorff dimension of F (see [18, Section 1.5]), $\kappa = \frac{\log 2}{\log 3}$.

Theorem 3.18 *Let $E = \{e^{it} : t \in F\}$. The function $S(z) = e^{-\frac{1+z}{1-z}}$ is cyclic in $\mathcal{B}^{\infty,0}_{\Lambda_\alpha,E}$ if and only if*

$$\alpha \le \frac{1}{1-\frac{\kappa}{2}}.$$

Proof. The set E is of zero measure ; all the complementary arcs are short or intermediate. For simplicity, we pass to $F \subset [0,1]$. For every $N \ge 1$ we have 2^N complementary intervals of length 3^{-N}. In every interval $[3^{s-N}, 2 \cdot 3^{s-N}]$ we have 2^s of such intervals, $0 \le s < N$. They are short for $3^{-s}(N-s)^{\alpha/2} \lesssim 1$ and intermediate for $3^{-s}(N-s)^{\alpha/2} \gtrsim 1$.

The sum for the intermediate intervals in (3.24) is

$$\sum_{N\ge1} \sum_{s\ge0,\,3^{-s}(N-s)^{\alpha/2}\gtrsim1} \sum_{I_n=[a_n,b_n]\subset[3^{s-N},2\cdot3^{s-N}]} \log^\alpha \frac{1}{b_n} \log\Big[\frac{b_n-a_n}{b_n}\log^{\alpha/2}\frac{1}{b_n}\Big]$$
$$\asymp \sum_{N\ge1} \sum_{s\ge0,\,3^{-s}(N-s)^{\alpha/2}\gtrsim1} \frac{\log^+(3^{-s}(N-s)^{\alpha/2})}{(N-s)^\alpha} \cdot 2^s \asymp \sum_{N\ge1} \frac{N^{\alpha\kappa/2}}{N^\alpha};$$

the latter series diverges if and only if $\alpha(1-\frac{\kappa}{2}) \le 1$.

The integral for the short intervals in (3.24) is

$$\sum_{N\ge1} \sum_{s\ge0,\,3^{-s}(N-s)^{\alpha/2}\lesssim1} \sum_{I_n=[a_n,b_n]\subset[3^{s-N},2\cdot3^{s-N}]} \int_{a_n}^{b_n} \frac{dt}{tw(t)}$$
$$\asymp \sum_{N\ge1} \sum_{s\ge0,\,3^{-s}(N-s)^{\alpha/2}\lesssim1} \sum_{I_n=[a_n,b_n]\subset[3^{s-N},2\cdot3^{s-N}]} \frac{1}{(N-s)^{\alpha/2}} \log\frac{b_n}{a_n}$$
$$\asymp \sum_{N\ge1} \sum_{s\ge0,\,3^{-s}(N-s)^{\alpha/2}\lesssim1} \frac{1}{N^{\alpha/2}} \cdot \frac{2^s}{3^s} \asymp \sum_{N\ge1} \frac{N^{(\alpha/2)(\kappa-1)}}{N^{\alpha/2}};$$

the latter series diverges if and only if $\alpha(1-\frac{\kappa}{2}) \le 1$. □

Since the Cantor set F is self-similar, we get the same result for every shift of E : $E_x = \{e^{i(y-x)} : e^{iy} \in E\}$, $e^{ix} \in E$. On the other hand, it looks difficult to characterize the threshold value of α in terms of (the local behavior near the point 1) for general sets E.

Chapitre 4

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